Fractals and Percolation

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Article Outline

Glossary

Percolation In the traditional meaning, percolation concerns the movement and filtering of fluids through porous materials. In this chapter, percolation is the subject of physical and mathematical models of porous media that describe the formation of a long-range connectivity in random systems and phase transitions. The most common percolation model is a lattice, where each site is occupied randomly with a probability p or empty with probability $1-p$. At low p values, there is no connectivity between the edges of the lattice. Above some concentration p_c, the *percolation threshold*, connectivity appears between the edges. Percolation represents a geometric critical phenomena where p is the analogue of temperature in thermal phase transitions.

Fractal A fractal is a structure which can be subdivided into parts, where the shape of each part is similar to that of the original structure. This property of fractals is called self-similarity, and it was first recognized by G.C. Lichtenberg more than 200 years ago. Random fractals represent models for a large variety of structures in nature, among them porous media, colloids, aggregates, flashes, etc. The concepts of self-similarity and fractal dimensions are used to characterize percolation clusters. Self-similarity is strongly related to renormalization properties used in critical phenomena, in general, and in percolation phase transition properties.

Definition of the Subject

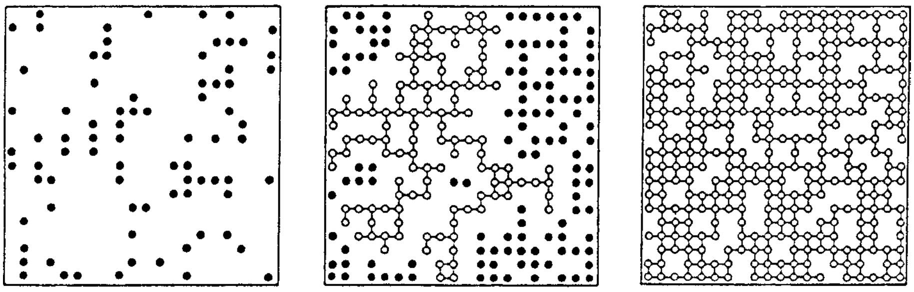
Percolation theory is useful for characterizing many disordered systems. Percolation is a pure random process of choosing sites to be randomly occupied or empty with certain probabilities. However, the structures obtained in such processes have a reach structure related to fractals. The structural properties of percolation clusters have become clearer thanks to the development of fractal geometry since the 1980s.

Introduction

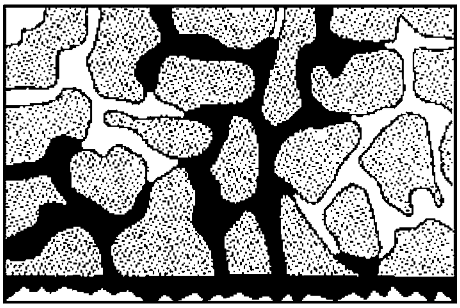
Percolation represents the simplest model of a phase transition [1,8,13,14,26,27,30,48,49,61,64,65,68]. Assume a regular lattice (grid) where each site (or bond) is occupied with probability p or empty with probability $1-p$. At a critical threshold, p_c, a long-range connectivity first appears: p_c is called the percolation threshold (see Fig. 1). Occupied and empty sites (or bonds) may stand for very different physical properties. For example, occupied sites may represent electrical conductors, empty sites may represent insulators, and electrical current may flow only through nearest-neighbor conducting sites. Below p_c, the grid represents an isolator since there is no conducting path between two adjacent bars of the lattice, while above p_c, conducting paths start to occur and the grid becomes a conductor. One can also consider percolation as a model for liquid filtration (i. e., invasion *percolation* (see Fig. 2), which is the source of this terminology) through porous media.

A possible application of bond percolation in chemistry is the polymerization process [25,31,44], where small branching molecules can form large molecules by activating more and more bonds between them. If the activation probability p is above the critical concentration, a network of chemical bonds spanning the whole system can be formed, while below p_c only macromolecules of finite size can be generated. This process is called a *sol-gel* transition. An example of this *gelation* process is the boiling of an egg, which at room temperature is liquid but, upon heating, becomes a solid-like *gel*.

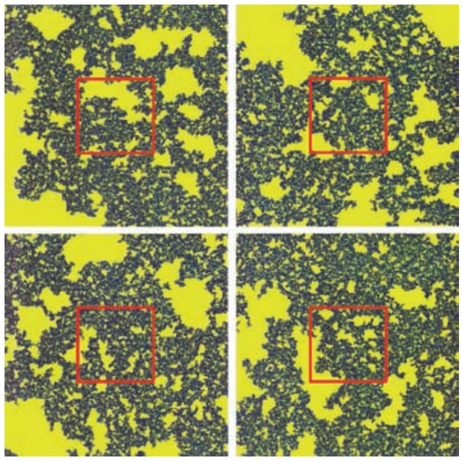
An example from biology concerns the spreading of an epidemic [35]. In its simplest form, an epidemic starts with one sick individual which can infect its nearest neighbors with probability p in one time step. After one time step, it dies, and the infected neighbors in turn can infect their (so far) uninfected neighbors, and the process is continued. Here the critical concentration separates a phase at low p where the epidemic always dies out after a finite number of time steps, from a phase where the epidemic can continue forever. The same process can be used as a model for



Fractals and Percolation, Figure 1
Square lattice of size 20 × 20. Sites have been randomly occupied with probability p ($p = 0.20, 0.59, 0.80$). Sites belonging to finite clusters are marked by *full circles*, while sites on the infinite cluster are marked by *open circles*



Fractals and Percolation, Figure 2
Invasion percolation through porous media



Fractals and Percolation, Figure 3
Self-similarity of the *random* percolation cluster at the critical concentration; courtesy of M. Meyer

forest fires [52,60,64,71], with the infection probability replaced by the probability that a burning tree can ignite its nearest-neighbor trees in the next time step. In addition to these simple examples, percolation concepts have been found useful for describing a large number of disordered systems in physics and chemistry.

The first study introducing the concept of percolation is attributable to Flory and Stockmayer about 65 years ago, when studying the gelation process [32]. The name percolation was proposed by Broadbent and Hammersley in 1957 when they were studying the spreading of fluids in random media [15]. They also introduced relevant geometrical and probabilistic concepts. The developments of phase transition theory in the following years, in particular the series expansion method by Domb [27] and renormalization group theory by Wilson, Fisher and Kadanoff [51,65], very much stimulated research activities into the geometric percolation transition.

At the percolation threshold, the conducting (as well as insulating) clusters are self-similar (see Fig. 3) and, therefore, can be described by fractal geometry [53], where various fractal dimensions are introduced to quantify the clusters and their physical properties.

Percolation

As above (see Sect. "Introduction"), consider a square lattice, where each site is occupied randomly with probability p (see Fig. 1). For simplicity, let us assume that the occupied sites are electrical conductors and the empty sites represent insulators. At low concentration p, the occupied sites either are isolated or form small clusters (Fig. 1a). Two occupied sites belong to the same cluster if they are connected by a path of nearest-neighbor occupied sites and a current can flow between them. When p is increased, the average size of the clusters increases. At a critical concentration p_c (also called the *percolation threshold*), a large cluster appears which connects opposite edges of the lattice (Fig. 1). This cluster is called the *infinite* cluster, since its size diverges when the size of the lattice is increased to infinity. When p is increased further, the density of the infinite cluster increases, since more and more sites become part of the infinite cluster, and the average size of the *finite* clusters decreases (Fig. 1c).

The *percolation threshold* separates two different phases and, therefore, the *percolation transition* is a *geometrical phase transition*, which is characterized by the geometric features of large clusters in the neighborhood of p_c. At low values of p, only small clusters of occupied sites exist. When the concentration p is increased, the average size of the clusters increases. At the critical concentration p_c, a large cluster appears which connects opposite edges of the lattice. Accordingly, the average size of the *finite* clusters which do not belong to the infinite cluster decreases. At $p = 1$, trivially, all sites belong to the infinite cluster.

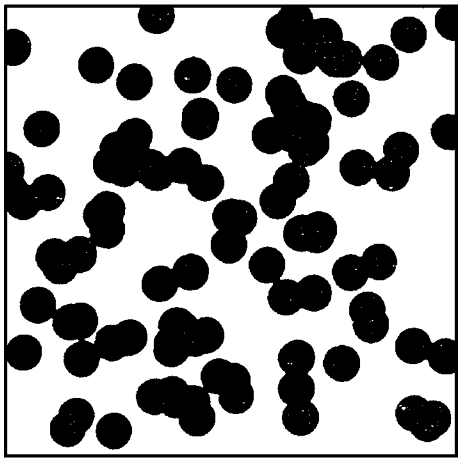
Similar to *site percolation*, it is possible to consider *bond percolation* when the bonds between sites are randomly occupied. An example of bond percolation in physics is a *random resistor network*, where the metallic wires in a regular network are cut at random. If sites are occupied with probability p *and* bonds are occupied with probability q, we speak of *site-bond percolation*. Two occupied sites belong to the same cluster if they are connected by a path of nearest-neighbor occupied sites with occupied bonds in between.

The definitions of site and bond percolation on a square lattice can easily be generalized to any lattice in d-dimensions. In general, in a given lattice, a bond has more nearest neighbors than a site. Thus, large clusters of bonds can be formed more effectively than large clusters of sites, and therefore, on a given lattice, the percolation threshold for bonds is smaller than the percolation threshold for sites (see Table 1).

A natural example of percolation, is *continuum percolation*, where the positions of the two components of a random mixture are not restricted to the discrete sites of a regular lattice [9,73]. As a simple example, consider a sheet of conductive material, with circular holes punched randomly in it (Swiss cheese model, see Fig. 4). The relevant quantity now is the fraction p of remaining conductive material.

Fractals and Percolation, Table 1
Percolation thresholds for the Cayley tree and several two- and three-dimensional lattices (see Refs. [8,14,41,64,75] and references therein)

Lattice	Percolation of	
	Sites	Bonds
Triangular	1/2	$2\sin(\pi/18)$
Square	0.5927460	1/2
Honeycomb	0.6962	$1 - 2\sin(\pi/18)$
Face Centered Cubic	0.198	0.119
Body Centered Cubic	0.245	0.1803
Simple Cubic (1^{st}nn)	0.31161	0.248814
Simple Cubic (2^{nd}nn)	0.137	–
Simple Cubic (3^{rd}nn)	0.097	–
Cayley Tree	$1/(z-1)$	$1/(z-1)$

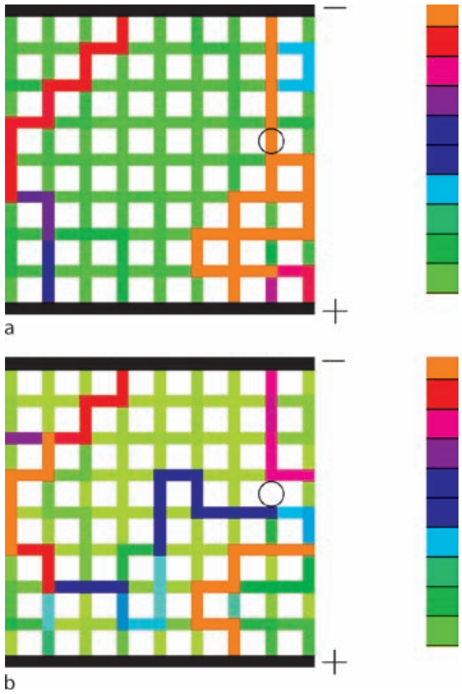


Fractals and Percolation, Figure 4
Continuum percolation: Swiss cheese model

Hopping Percolation

Above, we have discussed traditional percolation with only two values of local conductivities, 0 and 1 (insulator–conductor) or ∞ and 1 (superconductor–normal conductor). However, quantum systems should be treated by hopping conductivity, which can be described by an exponential function representing the local conductivities (between ith and jth sites): $\sigma_{ij} \sim \exp\left(-\kappa x_{ij}\right)$. Here κ can be interpreted as the dimensionless mean hopping distance or as the degree of disorder (the smaller the density of the deposited grains, the larger κ becomes), and x_{ij} is a random number taken from a uniform distribution in the range (0,1) [70].

In contrast to the traditional bond (or site) percolation model, in which the system is either a metal or an insulator, in the hopping percolation model the system always conducts some current. However, there are two regimes of such percolation [70]. A regime with many conducting paths which is not sensitive to the removal of a single bond (*weak disorder* $L/\kappa^{\nu} > 1$, where L is size of the system and ν is percolation critical exponent) and a regime

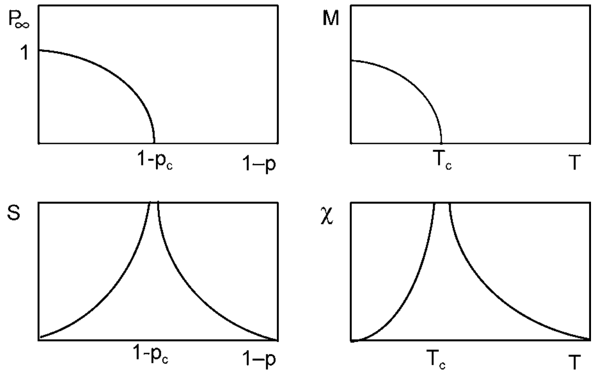


Fractals and Percolation, Figure 5
A color density plot of the current distribution in a bond-percolating lattice for which voltage is applied in the vertical direction for strong disorder with $\kappa = 10$. The current between the sites is shown by the different colors (*orange* corresponds to the highest value, *green* to the lowest). a The location of the resistor, on which the value of the local current is maximal, is shown by a *circle*. b The current distribution after removing the above resistor. This removal results in a significant change of the current trajectories

with a single or only a few dominating conducting paths which is very sensitive to the removal of a specific single bond with the highest current (*strong disorder* $L/\kappa^{\nu} \ll 1$). In the strong disorder regime, the trajectories along which the highest current flows (analogous to the spanning cluster at criticality in the traditional percolation network, see Fig. 5) can be distinguished and a single bond can determine the transport properties of the entire macroscopic system.

Percolation as a Critical Phenomenon

In percolation, the concentration p of occupied sites plays the same role as temperature in thermal phase transitions. The percolation transition is a geometrical phase transition where the critical concentration p_c separates a phase of finite clusters ($p < p_c$) from a phase where an infinite cluster is present ($p > p_c$).



Fractals and Percolation, Figure 6
P_∞ and S compared with magnetization M and susceptibility χ

An important quantity is the probability P_∞ that a site (or a bond) belongs to the infinite cluster. For $p < p_c$, only finite clusters exist, and $P_\infty = 0$. For $p > p_c$, P_∞ increases with p by a power law

$$P_\infty \sim (p - p_c)^{\beta} . \qquad (1)$$

P_∞ can be identified as the *order parameter* similar to magnetization, $m(T) \sim (T_c - T)^{\beta}$, in magnetic materials. With decreasing temperature, T, more elementary magnetic moments (spins) become aligned in the same direction, and the system becomes more ordered.

The linear size of the *finite* clusters, below and above p_c, is characterized by *correlation length* ξ. Correlation length is defined as the mean distance between two sites on the same finite cluster. When p approaches p_c, ξ increases as

$$\xi \sim |p - p_c|^{-\nu} , \qquad (2)$$

with the same exponent ν below and above the threshold. The mean number of sites (mass) of a finite cluster also diverges,

$$S \sim |p - p_c|^{-\gamma} , \qquad (3)$$

again with the same exponent γ above and below p_c. Analogous to S in magnetic systems is the susceptibility χ (see Fig. 6 and Table 2).

The exponents β, ν, and γ describe the critical behavior of typical quantities associated with the percolation transition, and are called *critical exponents*. The exponents are universal and do not depend on the structural details of the lattice (e. g., square or triangular) nor on the type of percolation (site, bond, or continuum), but depend only on the dimension d of the lattice.

Fractals and Percolation, Table 2
Exact and best estimate values for the critical exponents for percolation (see Refs. [8,14,41,64] and references therein)

Percolation	$d=2$	$d=3$	$d \geq 6$
Order parameter P_∞: β	5/36	0.417 ± 0.003	1
Correlation length ξ: ν	4/3	0.875 ± 0.008	1/2
Mean cluster size S: γ	43/18	1.795 ± 0.005	1

This universality property is a general feature of phase transitions, where the order parameter vanishes continuously at the critical point (second order phase transition).

In Table 2, the values of the critical exponents β, ν, and γ for percolation in two, three, and six dimensions. The exponents considered here describe the geometrical properties of the percolation transition. The physical properties associated with this transition also show power-law behavior near p_c and are characterized by critical exponents. Examples include the conductivity in a random resistor or random superconducting network and the spreading velocity of an epidemic disease near the critical infection probability. It is believed that the "dynamical" exponents cannot be generally related to the geometric exponents discussed above.

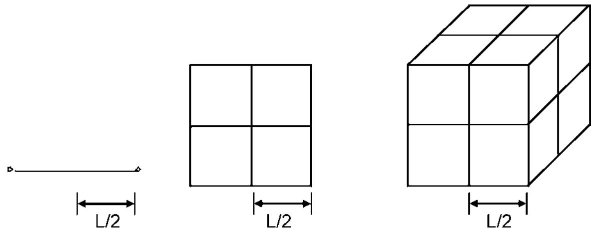
Note that all quantities described above are defined in the thermodynamic limit of large systems. In a finite system, for example, P_∞, is not strictly zero below p_c.

Percolation Clusters as Fractals

As first noticed by Stanley [66], the structure of percolation clusters (when the length scale is smaller than ξ) can be well described by the fractal concept [53]. Fractal geometry is a mathematical tool for dealing with complex structures that have no characteristic length scale. Scale-invariant systems are usually characterized by noninteger ("fractal") dimensions. This terminology is associated with B. Mandelbrot [53] (though some notion of noninteger dimensions and several basic properties of fractal objects were studied earlier by G. Cantor, G. Peano, D. Hilbert, H. von Koch, W. Sierpinski, G. Julia, F. Hausdorff, C. F. Gauss, and A. Dürer).

Fractal Dimension d_f

In regular systems (with uniform density) such as long wires, large thin plates, or large filled cubes, the dimension d characterizes how the mass $M(L)$ changes with the linear size L of the system. If we consider a smaller part of a system of linear size bL ($b < 1$), then $M(bL)$ is decreased by a factor of b^d, i. e.,



Fractals and Percolation, Figure 7
Examples of regular systems with dimensions $d = 1$, $d = 2$, and $d = 3$

$$M(bL) = b^d M(L) . \qquad (4)$$

The solution of the functional Eq. (4) is simply $M(L) = AL^d$. For a long wire, mass changes linearly with b, i. e., $d = 1$. For the thin plates, we obtain $d = 2$, and for cubes $d = 3$; see Fig. 7.

Mandelbrot coined the name "fractal dimension", and those objects described by a fractal dimension are called fractals. Thus, to include fractal structures, (4) we can generalize

$$M(bL) = b^{d_f} M(L) , \qquad (5)$$

and

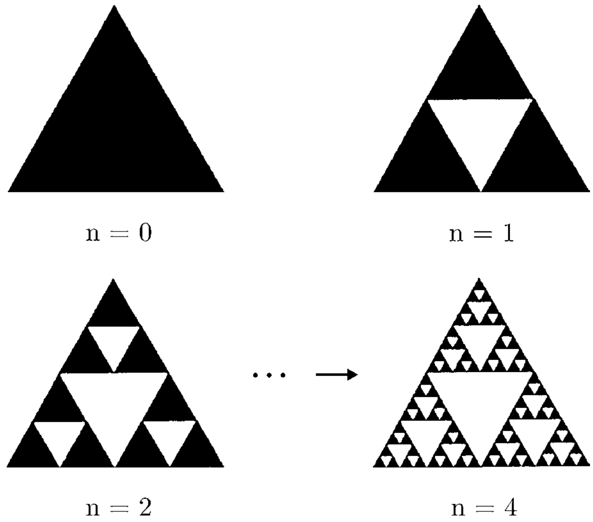
$$M(L) = AL^{d_f} , \qquad (6)$$

where d_f is the fractal dimension and can be a noninteger. Below, we present two examples of dealing with d_f: (i) the deterministic Sierpinski Gasket and (ii) random percolation clusters and criticality.

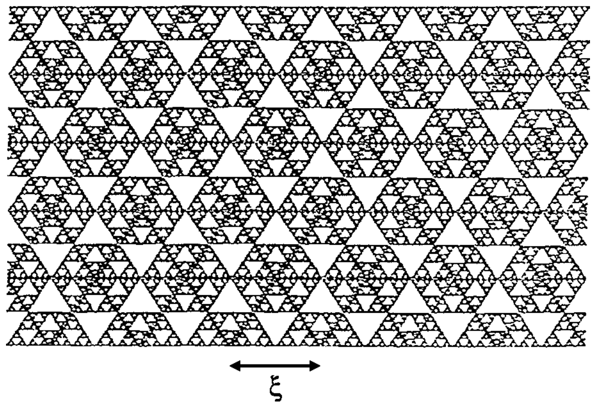
Sierpinski Gasket This fractal is generated by dividing a full triangle into four smaller triangles and removing the central triangle (see Fig. 8). In subsequent iterations, this procedure is repeated by dividing each of the remaining triangles into four smaller triangles and removing the central triangles. To obtain the fractal dimension, we consider the mass of the gasket within a linear size L and compare it with the mass within $\frac{1}{2}L$. Since $M(\frac{1}{2}L) = \frac{1}{3}M(L)$, we have $d_f = \log 3/\log 2 \cong 1.585$.

Percolation Fractal

We assume that at p_c ($\xi = \infty$) the clusters are fractals. Thus for $p > p_c$, we expect length scales smaller than ξ to have critical properties and therefore a fractal structure. For length scales larger than ξ, one expects a homogeneous



Fractals and Percolation, Figure 8
2D Sierpinski gasket. Generation and self-similarity



Fractals and Percolation, Figure 9
Lattice composed of Sierpinski gasket cells of size ξ

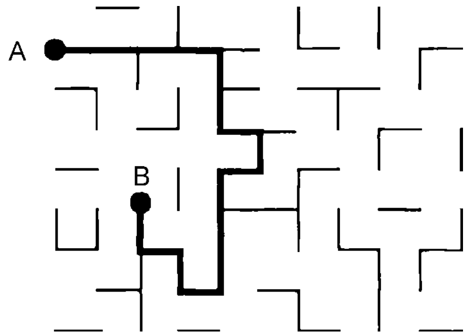
system which is composed of many unit cells of size ξ:

$$M(r) \sim \begin{cases} r^{d_f}, & r \ll \xi, \\ r^d, & r \gg \xi. \end{cases} \tag{7}$$

For a demonstration of this feature see Fig. 9.

One can relate the fractal dimension d_f of percolation clusters to the exponents β and ν. The probability that an arbitrary site within a circle of radius r smaller than ξ belongs to the infinite cluster, is the ratio between the number of sites on the infinite cluster and the total number of sites,

$$P_\infty \sim r^{d_f}/r^d, \qquad r < \xi. \tag{8}$$



Fractals and Percolation, Figure 10
Shortest path between two sites A and B on a percolation cluster

This equation is certainly correct for $r = a\xi$, where a is an arbitrary constant smaller than 1. Substituting $r = a\xi$ in (8) yields $P_\infty \sim \xi^{d_f}/\xi^d$. Both sides are powers of $p - p_c$. Substituting Eqs. (1) and (2) into the latter one obtains [8,28,41,49,64],

$$d_f = d - \beta/\nu. \tag{9}$$

Thus, the fractal dimension of the infinite cluster at p_c is not a new independent exponent but depends on β and ν. Since β and ν are universal exponents, d_f is also universal.

Shortest Path Dimensions, d_{min} and d_ℓ

The fractal dimension, however, is not sufficient to fully characterize a percolation cluster, since two clusters with very different topologies may have the same fractal dimension d_f. As an additional characterization of a fractal, one can consider, e.g., the shortest path between two arbitrary sites A and B on the cluster (see Figs. 10–11 [3,16,35,42,55,58].

The structure formed by the sites of this path is also self-similar and is described by a fractal dimension d_{min} [46,67]. Accordingly, the length ℓ of the path, which is often called the "chemical distance", scales with the "Euclidean distance" r between A and B as

$$\ell \sim r^{d_{min}}. \tag{10}$$

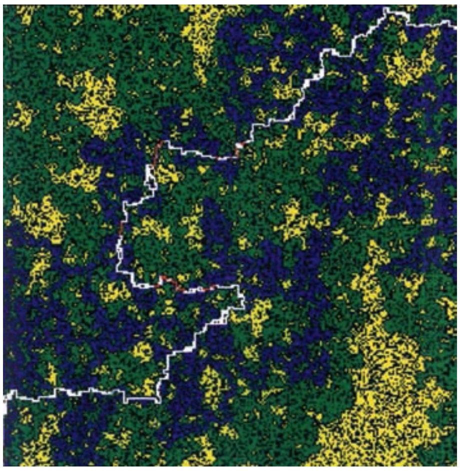
The inverse relation

$$r \sim \ell^{1/d_{min}} \equiv \ell^{\tilde{\nu}} \tag{11}$$

tells how r scales with ℓ.

Closely related to d_{min} and d_f is the "chemical" dimension d_ℓ, which describes how the cluster mass M within the chemical distance ℓ from a given site scales with ℓ,

$$M(\ell) \sim \ell^{d_\ell}. \tag{12}$$



Fractals and Percolation, Figure 11
Percolation system at critical concentration in a 510 × 510 square lattice. The finite clusters are in *yellow*. Substructures of the infinite percolation cluster are shown in different colors: the shortest path between two points at opposite sites of the system is shown in *white*, the single connected sites ("red" sites) in *red*, the loops in blue and the dangling ends in *green*; courtesy of S. Schwarzer

While the fractal dimension d_f characterizes how the mass of a cluster scales with the "Euclidean" distance r, the graph dimension d_ℓ characterizes how the mass scales with the chemical distance ℓ. Combining Eqs. (7), (10) and (12) we obtain the relation between $d_{\min}$, d_ℓ and d_f

$$d_\ell = d_f / d_{\min} \,. \tag{13}$$

To measure d_f, an arbitrary site is chosen on the cluster and one determines the number $M(r)$ of sites within a distance r from this site. To measure d_ℓ, an arbitrary site is chosen on the cluster at criticality and one determines the number $M(\ell)$ of sites which are connected to this site by a shortest path with length less than or equal to ℓ. Finally, to measure $d_{\min}$, two arbitrary sites are chosen on the cluster and one determines the length $\ell(r)$ of the shortest path connecting them. As for $M(r)$, averages must be performed for $M(\ell)$ and $r(\ell)$ over many realizations. In regular "Euclidean" lattices, both d_ℓ and d_f coincide with the Euclidean space dimension d and $d_{\min} = 1$.

The chemical dimension d_ℓ (or $d_{\min} = 1/\tilde{\nu}$) is an important tool for distinguishing between different fractal structures which may have a similar fractal dimension. In $d = 3$, for example, DLA (diffusion limited aggregation) clusters and percolation clusters have approximately the same fractal dimension $d_f \cong 2.5$, but have different $\tilde{\nu}$: $\tilde{\nu} = 1$ for DLA [54] but $\tilde{\nu} \cong 0.73$ for percolation [36,46,63,69].

While d_f has been related to the (known) critical exponents, (9), no such relation has been found for d_{min} or d_ℓ. The values of d_ℓ or $d_{\min}$ are known only from approximate methods, mainly numerical simulations (see also Refs. [17,37,57,76]).

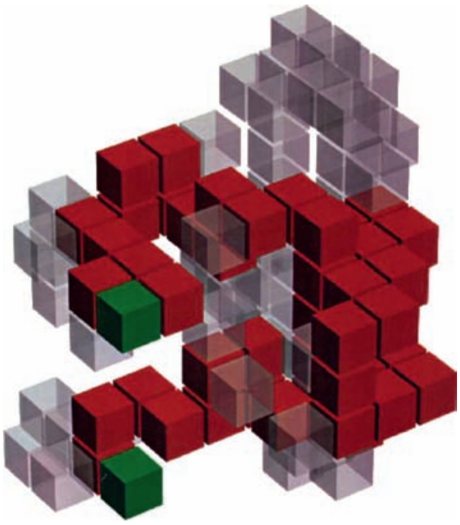
Fractal Substructures

The fractal dimensions d_f and d_ℓ are not the only exponents characterizing a percolation cluster at p_c. A percolation cluster is composed of several fractal sub-structures, which are described by other exponents. Imagine applying a voltage difference between two sites at opposite edges of a metallic percolation cluster: The *backbone* of the cluster consists of those sites (or bonds) which carry the electric current. The *dangling ends* are those parts of the cluster which carry no current and are connected to the backbone by a single site only. The *red bonds* (or singly connected bonds) [22,66] are those bonds that carry the total current; when they are cut the current flow stops. In analogy to red bonds we can define anti-red bonds [34]. If an anti-red bond is added to a nonconducting percolation system below p_c, the current will be able to flow in the system. The *blobs*, finally, are those parts of the backbone that remain after the red bonds have been removed.

Further fractal substructures of the cluster are the *external perimeter* (which is also called the *hull*), the *skeleton* and the *elastic backbone*. The hull consists of those sites of the cluster which are adjacent to empty sites and are connected with infinity via empty sites. In contrast, the *total* perimeter also includes the holes in the cluster. The external perimeter is an important model for random fractal interfaces. The skeleton is defined as the union of *all* shortest paths from a given site to all sites at a chemical distance ℓ [43]. The elastic backbone is the union of all shortest paths between two sites [47].

The fractal dimension d_B of the backbone is smaller than the fractal dimension d_f of the cluster (see Table 3). This reflects the fact that most of the mass of the cluster is concentrated in the dangling ends, which is seen clearly in Fig. 12. The value of the fractal dimension of the backbone is known only from numerical simulations [45,59]. Note, also, that the graph dimension d_ℓ^B of the backbone is smaller than that of percolation. In contrast, $\tilde{\nu}$ is the *same* for both backbone and percolation cluster, indicating the more universal nature of $\tilde{\nu}$. This can be understood by recalling that every two sites on a percolation cluster are located on the corresponding backbone.

The fractal dimensions of the red bonds d_{red} and the hull d_h are known from exact analytical arguments. It has been proven by Coniglio [22,23] that the mean number of



Fractals and Percolation, Figure 12
Percolation cluster at the critical concentration in a simple cubic lattice. The backbone between two (*green*) cluster sites is shown in *red*, the *gray* sites represent the dangling ends; courtesy of M. Porto

red bonds varies with p as

$$n_{\text{red}} \sim (p - p_c)^{-1} \sim \xi^{1/\nu} \sim r^{1/\nu} , \qquad (14)$$

and the fractal dimension of the red bonds is therefore $d_{\text{red}} = 1/\nu$. The fractal dimension of the skeleton is very close to $d_{\min} = 1/\tilde{\nu}$, supporting the assumption that percolation clusters at criticality are finitely ramified [43].

The hull of the cluster in $d = 2$ has the fractal dimension $d_h = 7/4$, which was first found numerically by Sapoval, Rosso, and Gouyet [63] and proven rigorously by Saleur and Duplantier [62]. If the hull is defined slightly differently and next-nearest neighbors of the perimeter are regarded as connected, many "fjords" are removed from the hull. According to Grossmann and Aharony [38], the fractal dimension of this modified hull is close to 4/3, the fractal dimension of self-avoiding random walks in $d = 2$. In three dimensions, in contrast, the mass of the hull seems to be proportional to the mass of the cluster, and both have the same fractal dimension.

In Table 3 we summarize the values of the fractal dimension d_f and the graph dimension d_ℓ of the percolation cluster and its fractal substructures.

Fractals and Percolation, Table 3
Fractal dimensions of the substructures composing percolation clusters (see Ref. [8,14,41,59,64] and references therein). For fractal dimensions in $d = 4$ and $d = 5$ see Ref. [57]

Fractal dimensions	Space dimension		
	$d = 2$	$d = 3$	$d \geq 6$
d_f	91/48	2.524 ± 0.008	4
d_ℓ	1.678 ± 0.005	1.84 ± 0.02	2
$d_{\min}$	1.13 ± 0.004	1.374 ± 0.004	2
d_{red}	3/4	1.143 ± 0.01	2
d_{h}	7/4	2.548 ± 0.014	4
d_{B}	1.64 ± 0.02	1.87 ± 0.04	2
d_ℓ^{B}	1.43 ± 0.02	1.34 ± 0.03	1

Anomalous Transport on Percolation Clusters: Diffusion and Conductivity

Due to self-similarity, transport quantities are significantly modified on fractal substrates. This can be seen in two representative examples:

(1) total resistance or the conductivity,
(2) mean square displacement and the probability density of random walks.

Consider a metallic network of size L^d. At opposite faces of the network are metallic bars with a voltage difference between them.

If we vary the linear size L of the system, the total resistance R varies as

$$R \sim \sigma^{-1} L / L^{d-1} , \qquad (15)$$

where $\sigma \sim L^0 = \text{const}$ is the conductivity (inverse to the resistivity, $\sigma = \rho^{-1}$) of the metal. Since σ does not depend on L, (15) states that the total resistance of the network depends on its linear size L via the power law $R \sim L^{2-d} \equiv L^{\tilde{\zeta}}$, which defines the resistance exponent $\tilde{\zeta}$, here $\tilde{\zeta} = 2 - d$.

The idea that transport properties of percolation systems can be efficiently studied by means of diffusion was suggested by de Gennes [24] (see also Kopelman [50]). The diffusion process can be modeled by random walkers, which can jump randomly between nearest-neighbor occupied sites in the lattice. For such a random walker moving in a disordered environment, including bottlenecks, loops, and dead ends, de Gennes coined the term *ant in the labyrinth*.

By calculating the mean square displacement of the walker, one obtains the diffusion constant, which according to Einstein is proportional to dc conductivity. Not only are the conductivity and diffusion exponents above p_c related; also related are the exponents characterizing the size dependence of the dc conductivity and the time dependence of the mean square displacement of the random walker. Since it is numerically more efficient to calculate the relevant transport quantities by simulating random walks than to determine the conductivity directly from Kirchhoff's equations, the study of random walks has improved our knowledge not only of diffusion but also of the transport process in percolation in general [5,6,10,12,18,39,40,56,74].

Due to the presence of large holes, bottlenecks, and dangling ends in the fractal, the motion of a random walker is slowed down. Fick's law for the mean square displacement ($\langle r^2(t)\rangle = a^2 t$) is no longer valid. Instead, the mean square displacement is described by a more general power law,

$$\langle r^2(t)\rangle \sim t^{2/d_w} , \tag{16}$$

where the new exponent d_w ("diffusion exponent" or "fractal dimension of the random walk") is always greater than 2.

Both the resistance exponent $\tilde{\zeta}$ and the exponent d_w can be related by the Einstein equation

$$\sigma = (e^2 n/k_B T) D , \tag{17}$$

which relates the dc conductivity σ of the system to the diffusion constant $D = \lim_{t\to\infty}\langle r^2(t)\rangle/2dt$ of the random walk. In Eq. (17), e and n denote charge and density of the mobile particles, respectively.

Simple scaling arguments can now be used to relate d_w to $\tilde{\zeta}$ and $\tilde{\mu}$. Since n is proportional to the density of the substrate, $n \sim L^{d_f - d}$, the right-hand side of Eq. (17) is proportional to $L^{d_f-d} t^{2/d_w - 1}$. The left-hand side of Eq. (17) is proportional to $L^{-\tilde{\mu}}$. Since the time a random walker takes to travel a distance L scales as L^{d_w}, we find $L^{-\tilde{\mu}} \sim L^{d_f - d + 2 - d_w}$, from which the "Einstein relation" [4]

$$d_w = d_f - d + 2 + \tilde{\mu} = d_f + \tilde{\zeta} \tag{18}$$

follows. For example, for the Sierpinski gasket $d_f = \ln 3/\ln 2$, and $\tilde{\zeta} = \log(5/3)/\log(2/4)$, therefore $d_w = \ln 5/\ln 2$.

In general, determining d_w for random fractals is not easy. An exception is topologically linear fractal structures ($d_\ell = 1$), which can be considered as nonintersecting paths. Along a path (in ℓ-space), diffusion is normal and $\langle \ell^2(t)\rangle = t$. Since $\ell \sim r^{d_f}$, the mean square displacement in r-space scales as $\langle r^2\rangle \sim t^{1/d_f}$, leading to $d_w = 2d_f$ in this case. In percolation, d_w cannot be calculated exactly, but upper and lower bounds can be derived which are very close to each other in $d \geq 3$ dimensions. A good estimate is $d_w \cong 3d_f/2$ (Alexander–Orbach conjecture [4]).

The long-term behavior of the mean square displacement of a random walker on an infinite percolation cluster is characterized by the diffusion constant D. It is easy to see that D is related to the diffusion constant D' of the whole percolation system: above p_c, the dc conductivity of the percolation system increases as $\sigma \sim (p - p_c)^\mu$, so due to the Einstein relation, Eq. (17), the diffusion constant D' must also increase in this way. The mean square displacement (and hence D') is obtained by averaging over all possible starting points of a particle in the percolation system. It is clear that only those particles which start on the infinite cluster can travel from one side of the system to the other, and thus contribute to D'. Particles that start on a finite cluster cannot leave the cluster, and thus do not contribute to D'. Hence D' is related to D by $D' = DP_\infty$, implying

$$D \sim (p - p_c)^{\mu - \beta} \sim \xi^{-(\mu-\beta)/\nu} . \tag{19}$$

Combining (16) and (19), the mean square displacement on the infinite cluster can be written as [7,33,72]

$$\langle r^2(t)\rangle \sim \begin{cases} t^{2/d_w} & \text{if } t \ll t_\xi , \\ (p - p_c)^{\mu-\beta}\, t & \text{if } t \gg t_\xi , \end{cases} \tag{20}$$

where

$$t_\xi \sim \xi^{d_w} \tag{21}$$

describes the time scale the random walker needs, on average, to explore the fractal regime in the cluster. As $\xi \sim (p - p_c)^{-\nu}$ is the only length scale here, t_ξ is the only relevant time scale, and we can bridge the short time regime and the long time regime by a scaling function $f(t/t_\xi)$,

$$\langle r^2(t)\rangle = t^{2/d_w} f(t/t_\xi) . \tag{22}$$

To satisfy (20)–(21), we require $f(x) \sim x^0$ for $x \ll 1$ and $f(x) \sim x^{1-2/d_w}$ for $x \gg 1$. The first relation trivially satisfies (20)–(21). The second relation gives $D = \lim_{t\to\infty}\langle r^2(t)\rangle/2dt \sim t_\xi^{2/d_w - 1}$, which in connection with (19) and (21) yields a relation between d_w and μ [7, 33,72],

$$d_w = 2 + (\mu - \beta)/\nu . \tag{23}$$

Comparing (18) and (23) one can express the exponent $\tilde{\mu}$ by μ,

$$\tilde{\mu} = \mu/\nu . \tag{24}$$

Networks

Networks are defined as nodes connected by links, called graphs in mathematics. Many real world system can be describe as networks. Perhaps the best known example of a network is the Internet, where computers (nodes) around the globe are connected by cables (links) in such a way that an email message can travel from one computer to another by traveling along only a few links. Social relations between people can be represented by a social network [2]; nodes represent people and links represent their relations. One important property of a network is the "small world" phenomena. The shortest path (minimum number of hops) between any two nodes is very small, of the order of $\log N$ or smaller, where N is the number of nodes in the network [11,19,29]. The lattices discussed in Sect. "Percolation" are also networks, where the sites of the lattice are the nodes and the bonds represent the links. In this case, the number of links per node is fixed but, in general, the number of links per node can be taken from any degree distribution, $P(k)$. In lattices, due to spatial constraints, the distances between nodes is large and scales as $N^{1/d}$, where d is the dimension of the lattice. Since many networks have no spatial constraints, it follows that such networks can be regarded as embedded in infinite dimension, $d = \infty$, justifying a very small distance, of order $\log N$. In networks which are not constrained to geographical space, there is no Euclidean distance and the distance metric is only the shortest path ℓ defined in Sect. "Percolation Clusters as Fractals".

We will show in this chapter that ideas from percolation and fractals can be applied to obtain useful results in networks which are not embedded in space. The main difference compared to lattices is that the condition for percolation is no longer the spanning property, but rather the property of having a cluster containing something on the order CE2 of N nodes, where N is the total original number of nodes in the network. Such a component, if it exists, is termed the *giant component*. The condition for the existence of a giant component above the percolation threshold, and its absence below the threshold, applies also to lattices, and therefore can be regarded as more general than the spanning property. An interesting property of percolation, called *universality*, is that the behavior at and close to the critical point depends only on the dimensionality of the lattice, and not on the microscopic connection details of the lattice. This behavior is characterized by a set of critical exponents that are the same for all two-dimensional lattices, square, triangular or hexagonal, and for either site or bond percolation. However, a different set of critical exponents will be obtained for a lattice of another dimension. Furthermore, above some critical dimension ($d_c = 6$ for percolation in d-dimensional lattices), known as the upper critical dimension, the critical behavior remains the same for all $d \geq d_c$. This is due to the insignificance of loops in high dimensions, and thus usually allows for easy determination of the critical exponents for high dimensions, using the "infinite dimensional" or "mean field" approach. Erdős and Rényi (ER) studied an ensemble of networks with N nodes and $2M$ links that randomly connect pairs of nodes. They found that $p_c = 1/\langle k \rangle = N/2M$. Percolation on ER networks or on infinite dimensional lattices, as well as on Cayley trees, has the same critical exponents, due to the fact that their topology is the same and no spatial constraint is imposed on the networks. For ER networks, as for lattices, in $d \geq 6$ the size S of the percolation cluster at p_c, scales with N as, $S \sim N^{2/3}$ [11,20].

The value of 2/3 can be related to the upper critical dimension, $d_c = 6$, and to the fractal dimension of percolation clusters $d_f = 4$ for $d = 6$. Since $N = L^6$ and $S = L^4$, it follows that $S \sim N^{2/3}$.

In recent years it was realized [2] that $P(k)$ for many real networks is very broad and, in many cases, is best represented by a power law, $P(k) \sim k^{-\gamma}$. Networks with a power law degree distribution are called *scale free (SF) networks*. Heterogeneity of the degrees may affect critical behavior, even above the upper critical dimension. The heterogeneity of the degrees can be regarded as a breakdown of translational symmetry that exists in lattices, ER networks and Cayley trees. In these cases, each node has a typical number of neighbors, while in scale free networks the variation between node degrees is very large. A general result for p_c for any random network with a given degree distribution is [21]

$$p_c = \frac{1}{\kappa - 1}, \quad \kappa \equiv \frac{\langle k^2 \rangle}{\langle k \rangle}.$$

This result yields that for $\gamma < 3$, $\langle k^2 \rangle \to \infty$ and therefore $p_c \to 0$. That is, no finite percolation threshold exists. Thus, even if most of the nodes of the Internet are removed, those which are left can still communicate. This explains the puzzle of why viruses and worms stay a long time in the Internet even if many people use antivirus software. It also explains why in order to effectively immunize populations, one needs to immunize most of the people.

The percolation critical exponents for SF networks are still mean-field or infinite-dimensional in the sense of the insignificance of loops. However, they are different from those of the standard mean field percolation. Indeed, for scale free networks [2], the size of the spanning percolation cluster is [20],

$$S \sim \begin{cases} N^{(\gamma-2)/(\gamma-1)}, & 3 < \gamma < 4 \\ N^{2/3}, & \gamma > 4 \end{cases} \quad (25)$$

As shown above, for $\gamma < 3$, there is no percolation threshold, and therefore no spanning percolation cluster. Note that SF networks can be regarded as a generalization of ER networks, since for $\gamma > 4$ one obtains the ER network results.

Summary and Future Directions

The percolation problem and its numerous modifications can be useful in describing several physical, chemical, and biological processes, such as the spreading of epidemics or forest fires, gelation processes, and the invasion of water into oil in porous media, which is relevant for the process of recovering oil from porous rocks. In some cases, modification changes the universality class of a percolation. We begin with an example in which the universality class does not change. We showed that a random process such as percolation can lead naturally to fractal structures. This may be one of the reasons why fractals occur frequently in nature.

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