Strong-field magnetotransport of conducting composites with a columnar microstructure

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The transport properties of composite conductors with a columnar microstructure, and subject to a strong uniform magnetic field perpendicular to the columnar axis, are discussed with special emphasis on the strong-field limit. Asymptotic considerations provide detailed insight into the factors that govern the local current distribution, and hence the bulk effective Ohmic resistivity components. Particularly explicit asymptotic results are found in the case of periodic arrays of long, parallel inclusions that are either perfect insulators or perfect conductors, and are embedded in a conducting host that is a simple, free-electron-like conductor. In some configurations, simple closed form expressions are derived for the current distribution as well as for the bulk effective resistivities. Numerical calculations of those distributions and resistivities are also performed and compared to the asymptotic expressions. We thus achieve a detailed understanding of the rich phenomenology of strong-field magnetotransport in such systems, some of which was predicted earlier from numerical calculations and subsequently found in a recent experiment. [S0163-1829(99)06803-4]

I. INTRODUCTION

Not long ago, it was first predicted that composites made of a periodic array of inclusions embedded in a conducting host would exhibit a strongly anisotropic dependence of their Ohmic resistivity on a uniform applied magnetic field $B$, when that field is strong enough. Such behavior, which appears as soon as the Hall resistivity of the host exceeds its Ohmic resistivity, was recently observed for the first time in a thin semiconducting film, in which a periodic array of holes were etched, when a strong magnetic field was applied in the film plane. In this experiment, the microstructure was approximately columnar—the etched holes were shaped approximately as circular cylinders perpendicular to the film plane. Such periodic columnar microstructures are easier to fabricate than microstructures, which have a nontrivial three-dimensional periodicity. Interestingly, our earlier numerical studies showed that in columnar microstructures the magnitude of the magnetoresistance, as well as its anisotropy, are greater than in three-dimensional microstructures, given similar values for the lattice parameter and inclusion sizes. This observation has motivated some recent studies of magnetotransport in conducting composites with a periodic columnar microstructure. Those include a study based on symmetry considerations and numerical calculations, and computer simulations of a discrete network model.

We now present a detailed theoretical discussion of the magnetotransport properties of conducting composites with a columnar microstructure. In order to do this we exploit the two dimensional (2D) nature of the heterogeneity in an essential way. One of the consequences of this columnar symmetry is that it becomes possible to perform a duality transformation on the local electric field and current distributions. This is a nontrivial extension of the well-known duality transformation for 2D systems. It enables us to recast the classical magnetotransport problem in terms of a dual classical transport problem, where the (dual) conductivity tensor is symmetric, even though the original problem was characterized by a nonsymmetric resistivity tensor.

The results we get for the magnetotransport in periodic columnar microstructures include a positive magnetoresistance that sometimes never saturates with increasing $B$, and local distorted current distributions that sometimes become extremely simple when $|B| \to \infty$, and sometimes change drastically when the microstructure is changed in a continuous and seemingly harmless fashion. In many cases, closed form expressions for the asymptotic (i.e., large $B$) current distributions can be obtained from a simple, intuitive, physical consideration of the dual problem, for periodic arrays of inclusions that are either perfect insulators or perfect conductors. This leads to simple, sometimes even closed form, expressions for the asymptotic behavior of the bulk effective Ohmic resistivities of such materials. Thus we are able to make precise quantitative predictions for the asymptotic values of the transverse and longitudinal bulk effective magnetoresistivities. Moreover, we can predict general forms of asymptotic behavior even in cases where such closed form expressions are unobtainable. In particular, we find that in the case of a periodic array of insulating inclusions, the in-plane transverse magnetoresistivity $\rho_{\perp}^{(e)}$ essentially never saturates with increasing $B$. For large $B$ it is proportional to $B^2$, and it also has a strong dependence on the direction of $B$. By contrast, in the case of a periodic array of perfectly conducting inclusions, $\rho_{\perp}^{(e)}$ usually saturates with increasing $B$, except for a small set of directions of $B$. When $B$ is along one of those directions, which is always a low order lattice axis, then $\rho_{\perp}^{(e)} \propto B^4$ for large $B$. As a result of this, the angular profile of $\rho_{\perp}^{(e)}$ versus $B$ is even more anisotropic for large $B$ in the case of perfectly conducting inclusions than in the case of insulating inclusions.

The remainder of this article is organized as follows. In Sec. II we develop the theory necessary for a discussion of composite conductors with a columnar microstructure, subject to a uniform applied magnetic field. In Sec. III we discuss the asymptotic strong field magnetoresistance of some specific microstructures, usually where a periodic array of columnar inclusions is embedded in a simple, free-electron-
like host. The inclusions are taken to be either perfect insulators or perfect conductors. The discussion is based upon the theory developed in Sec. II, and also on extensive numerical computations that we performed, using methods developed before for such problems.\textsuperscript{1,9} Section IV provides a summary and discussion of the main results.

II. THEORY

The electrical transport on the microscale is assumed to be describable in terms of a local curl-free electric field \( \mathbf{E}(\mathbf{r}) = \nabla \phi(\mathbf{r}) \) and a local divergence-free current density \( \mathbf{J}(\mathbf{r}) \), which are related to each other linearly by means of a local resistivity tensor \( \hat{\rho}(\mathbf{r}) \) or a local conductivity tensor \( \hat{\sigma}(\mathbf{r}) = 1/\hat{\rho}(\mathbf{r}) \),

\[
\mathbf{J}(\mathbf{r}) = \hat{\sigma}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}).
\]

These fields are found by solving the usual equation

\[
\nabla \cdot \hat{\sigma} \cdot \nabla \phi = 0,
\]

with appropriate boundary conditions on \( \phi(\mathbf{r}) \). After that, the bulk effective resistivity and conductivity tensors \( \hat{\rho}_e \) and \( \hat{\sigma}_e \) can be calculated from

\[
\langle \mathbf{J} \rangle = \hat{\sigma}_e \cdot \langle \mathbf{E} \rangle, \quad \hat{\rho}_e = 1/\hat{\sigma}_e,
\]

where \( \langle \cdot \rangle \) denotes a volume average over the system. The bulk effective tensors \( \hat{\sigma}_e, \hat{\rho}_e \) characterize the macroscopic electrical response of the system, i.e., its transport behavior on length scales much larger than the typical microstructure or heterogeneity length scales, when we are interested only in some coarse grained volume average values of \( \mathbf{E} \) and \( \mathbf{J} \).

The antisymmetric part of \( \hat{\rho}(\mathbf{r}) \), i.e., the local Hall resistivity, is usually proportional to \( \mathbf{B} \), and continues to grow indefinitely with increasing \( \mathbf{B} \). By contrast, the symmetric part of \( \hat{\rho}(\mathbf{r}) \), i.e., the local Ohmic resistivity, usually tends to a finite limit when \( \mathbf{B} \to \infty \)—this is called saturation of the magnetoresistance. The symmetric part of \( \hat{\rho}_e \) can also be obtained by considering the volume average of the local rate of production of Joule heat \( W(\mathbf{r}) = \mathbf{J}(\mathbf{r}) \cdot \hat{\rho}(\mathbf{r}) \cdot \mathbf{J}(\mathbf{r}) \):

\[
\langle \mathbf{J} \rangle \cdot \hat{\rho}_e \cdot \langle \mathbf{J} \rangle = \langle \mathbf{J} \cdot \hat{\rho} \cdot \mathbf{J} \rangle.
\]

Note that only the symmetric parts of \( \hat{\rho} \) and \( \hat{\rho}_e \) actually appear in this equation. The average current density \( \langle \mathbf{J} \rangle \) is usually fixed by the boundary conditions—both in experiment and in a calculation. Therefore the symmetric part of \( \hat{\rho}_e \) can fail to saturate only if the local current-density distribution \( \mathbf{J}(\mathbf{r}) \) does not saturate as \( \mathbf{B} \to \infty \).

In true three-dimensional microstructures, \( \mathbf{J}(\mathbf{r}) \) apparently always saturates.\textsuperscript{10} In contrast, in columnar microstructures \( \mathbf{J}(\mathbf{r}) \) usually does not saturate as \( \mathbf{B} \to \infty \). This was already noted before, following results of numerical calculations of \( \hat{\rho}_e \)—see Fig. 13 of Ref. 1. In the following subsection, where we consider such microstructures, we shall explain how this comes about.

A. Infinite columnar microstructures

In such systems, the heterogeneity has a two-dimensional character (see Fig. 1). Nevertheless, the physical transport will usually have an essentially three-dimensional character; i.e., the fields \( \mathbf{E}(\mathbf{r}), \mathbf{J}(\mathbf{r}) \) will usually have nonzero components along the axis of columnar symmetry when \( \mathbf{B} \) has a nonzero component perpendicular to that axis. We will always choose the \( x \) axis to lie along that axis; therefore the microstructure is independent of \( x \). Consequently, if the boundary conditions are also independent of \( x \), (this requires that the system extend to \( \pm \infty \) along the \( x \) axis) then both \( \mathbf{E}(\mathbf{r}) \) and \( \mathbf{J}(\mathbf{r}) \) will actually depend only on \( y \) and \( z \). Moreover, it is easy to see that \( \nabla \times \mathbf{E} = 0 \) then entails that \( E_z \) is uniform everywhere.

This has some far reaching consequences. In order to exhibit those more clearly, we consider the special case where the resistivity tensor \( \hat{\rho}(y,z) \) has an isotropic, free-electron-like form everywhere, and the magnetic field \( \mathbf{B} \) lies along the \( z \) axis (i.e., we assume an in-plane magnetic field), namely,

\[
\hat{\rho} = \rho_0 \begin{pmatrix} 1 & -H & 0 \\ H & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

where the Ohmic resistivity \( \rho_0 \) is independent of \( \mathbf{B} \) and \( H = \mu |\mathbf{B}| = \omega_c \tau \) is the Hall-to-Ohmic resistivity ratio. [Note that \( \tau \) is the conductivity relaxation time, \( \omega_c = e|\mathbf{B}|/(mc) \) is the cyclotron frequency, \( \mu \) is the Hall mobility; the sign of \( \omega_c \) and \( \mu \) is the same as the sign of the charge \( e \) of the elementary charge carriers.]

In that case, the uniform electric-field component \( E_z \) satisfies the following equations:

\[
E_z = \rho_0 (J_z - H J_y) = \frac{\langle J_z \rangle - (H J_y)}{\langle 1/\rho_0 \rangle}.
\]

Eliminating \( E_z \), we get

\[
J_z = H J_y + \frac{1}{\langle 1/\rho_0 \rangle} \langle J_z - H J_y \rangle.
\]

Because the spatial heterogeneity of \( H J_y \) is usually different from that of \( 1/\rho_0 \), even if \( J_y \) saturates, \( J_z \) will usually not saturate, but instead its local magnitude will continue to grow as \( H \). In the wire-like system shown in Fig. 1, where the average current \( \langle \mathbf{J} \rangle \) lies in the \( y,z \)-plane and \( \langle J_z \rangle = 0 \), the local values of \( J_z \) continue to fluctuate from point to point in the \( y,z \)-plane with an amplitude that is proportional to \( H \).

In a two component composite medium made of perfectly insulating, parallel columnar inclusions (\( \rho_0 = \infty \)), embedded in a free-electron-like conducting host, the volume averages in Eq. (2.7) only involve the host subvolume. We thus get, when \( \langle J_y \rangle = 0 \),

\[
J_z = H_{\text{host}} \left( J_y - \frac{\langle J_y \rangle}{\rho_{\text{host}}} \right).
\]
in the host, where \( p_{\text{host}} = V_{\text{host}}/V_{\text{tot}} \) is the volume fraction occupied by the host. From this result it is clear that \( J_x \) will fluctuate in space like \( J_y \), but with an amplitude that increases indefinitely with \( |B| \). The only way to avoid this non-saturation of \( J_x \) is by having

\[
J_x = \langle J_x \rangle / p_{\text{host}} \tag{2.9}
\]

in the host, i.e., if \( J_y \) is uniform everywhere in the host sub-volume. The average rate of dissipation in such a system is given by

\[
\frac{\langle W \rangle}{\rho_{0\text{host}}} = H_x^2 \left( \frac{1}{\rho_{\text{host}}} \langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \right), \tag{2.10}
\]

where the first term on the right-hand side (r.h.s.) is strictly positive, unless Eq. (2.9) is satisfied, in which case that term vanishes (this can easily be shown by using the Cauchy-Schwarz inequality\(^1\)). The significance of this result is that, in such a system, the magnetoresistance will usually fail to saturate, increasing as \( B^2 \) for large \( B \).

In the opposite case, where the inclusions are perfect conductors \((\rho_0 = 0)\), \( E_x \) must vanish everywhere, therefore in the normal host we must have

\[
J_x = H_{\text{host}} J_y. \tag{2.11}
\]

Thus, unless \( J_y \) vanishes everywhere in the host, \( J_x \) will again fail to saturate. The average rate of dissipation is now given by

\[
\frac{\langle W \rangle}{\rho_{0\text{host}}} = (H_x^2 + 1) \langle J_y^{2\text{host}} \rangle + \langle J_z^{2\text{host}} \rangle, \tag{2.12}
\]

where \( \langle J_y^{2\text{host}} \rangle \), \( \langle J_z^{2\text{host}} \rangle \) denote volume averages where only currents in the host are included: The currents in the perfectly conducting inclusions do not contribute to the dissipation. Again, we may conclude that usually the magnetoresistance of such a system will not saturate at large \( B \).

**B. Duality in a system with columnar symmetry**

The duality transformation in such systems is an extension of the well-known duality transformation for two-dimensional systems, where the planar fields \( \mathbf{E}, \mathbf{J} \) are rotated in the plane by 90° to become the dual fields \( \mathbf{E}_D, \mathbf{J}_D \), respectively.\(^7\) In a columnar system, where \( \mathbf{E} \) and \( \mathbf{J} \) are three-dimensional fields, we perform a similar 90° rotation only of the \( y \) and \( z \) components of those fields, i.e., the in-plane components, leaving the columnar or \( x \) components intact. We then define the dual electric field \( \mathbf{E}_D \) and dual current density \( \mathbf{J}_D \) as follows:

\[
\mathbf{E}_D = (E_x, -\rho_0 J_z, \rho_0 J_y), \tag{2.13}
\]

\[
\mathbf{J}_D = (J_x, -E_z/\rho_0, E_y/\rho_0), \tag{2.14}
\]

where \( \rho_0 \) is some constant resistivity that is introduced in order to make the physical dimensions of the different components of \( \mathbf{E}_D \) and \( \mathbf{J}_D \) consistent. In the case of a two-component mixture where one component is either a perfect insulator or a perfect conductor, a natural choice for \( \rho_0 \) is the finite Ohmic resistivity of the normal conducting component. It is easy to see that \( \nabla \times \mathbf{E}_D = 0 \) and \( \nabla \cdot \mathbf{J}_D = 0 \), and also that

\[
\mathbf{J}_D = \hat{\mathbf{D}} \cdot \mathbf{E}_D, \tag{2.15}
\]

where the precise form of the dual conductivity tensor \( \hat{\mathbf{D}} \) can be worked out from \( \hat{\mathbf{D}} \) or \( \hat{\mathbf{D}} \), and is usually nonsymmetric even if \( \hat{\mathbf{D}} \) and \( \hat{\mathbf{D}} \) were symmetric to begin with. However, it is both remarkable and convenient that when \( \hat{\mathbf{D}} \) has the special nonsymmetric form of Eq. (2.5), then \( \hat{\mathbf{D}}^{\hat{\mathbf{D}}} \) and \( \hat{\mathbf{D}}^{\hat{\mathbf{D}}} \) are in fact symmetric (actually, the same would be true even if \( \rho_{zz} \) differed from \( \rho_{0z} \) in Eq. (2.5)). Thus, for a two-component mixture made of perfectly insulating columnar inclusions embedded in a free-electron-like conducting host, the conductivity and resistivity tensors of the dual problem are (we take \( \rho_{00} = \rho_0 \), as mentioned above)

\[
\hat{\mathbf{D}}^{\hat{\mathbf{D}}} = \frac{1}{\rho_0} \begin{pmatrix} 1 & 0 & H \\ 0 & 1 & 0 \\ H & 0 & 1 + H^2 \end{pmatrix}, \tag{2.16}
\]

\[
\hat{\mathbf{D}}^{\hat{\mathbf{D}}} = \rho_0 \begin{pmatrix} 1 + H^2 & 0 & -H \\ 0 & 1 & 0 \\ -H & 0 & 1 \end{pmatrix}, \tag{2.17}
\]

\[
\hat{\mathbf{D}}^{\hat{\mathbf{D}}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \infty \end{pmatrix}, \quad \hat{\mathbf{D}}^{\hat{\mathbf{D}}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \infty \end{pmatrix}. \tag{2.18}
\]

Because these tensors are symmetric, the dual problem is conceptually simpler than the original problem. It also has the usual variational property, namely, that the total rate of Joule heating is minimal for the correct physical distribution of \( \mathbf{E}_D(\mathbf{r}) \) and \( \mathbf{J}_D(\mathbf{r}) \). This is sometimes useful as a basis for approximation procedures, or in order to decide which of several approximate distributions is best. An example of such usage will be given in Sec. III D below.

In the opposite case where the inclusions are perfect conductors, the dual problem is characterized by transposing the previous expressions for \( \hat{\mathbf{D}}^{\hat{\mathbf{D}}} \) and \( \hat{\mathbf{D}}^{\hat{\mathbf{D}}} \),

\[
\hat{\mathbf{D}}^{\hat{\mathbf{D}}} = \begin{pmatrix} \infty & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \infty \end{pmatrix}, \quad \hat{\mathbf{D}}^{\hat{\mathbf{D}}} = \begin{pmatrix} \infty & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \infty \end{pmatrix}. \tag{2.19}
\]

The electric field \( \mathbf{E} \) must vanish inside such inclusions, therefore \( E_z = 0 \) everywhere.

The tensors \( \hat{\mathbf{D}}^{\hat{\mathbf{D}}} \), \( \hat{\mathbf{D}}^{\hat{\mathbf{D}}} \) can easily be diagonalized by choosing new coordinate axes where the \( x \) and \( z \) axes are rotated about the \( y \) axis by an angle \( \alpha \), where

\[
\tan \alpha = \frac{H}{2} = \left( \frac{H^2}{4} + 1 \right)^{1/2}. \tag{2.20}
\]

and the principal conductivities along the rotated axes are

\[
\frac{1}{\rho_0} \left[ 1 + \frac{H^2}{2} \pm H \left( \frac{H^2}{4} + 1 \right)^{1/2} \right]. \tag{2.21}
\]
When $|H| \gg 1$, the rotation angle is very small $\alpha \approx 1/|H|$, and the principal conductivities are, to leading order, $H^2/\rho_0$ and $1/(H^2 \rho_0)$. Thus, to leading order in $H$, the dual host is simply a very anisotropic conductor, with almost the same principal axes as the dual inclusions and the microstructure. As we shall see in Sec. III below, in periodic microstructures this often leads to simple current distributions for which closed form asymptotic expressions can be derived from simple physical considerations.

C. Finite thickness effects

Until now we have assumed that the composite columnar structure fills all space. This is not a serious restriction as far as lateral dimensions are concerned—it is easy to fabricate periodically microstructured films with a large number of unit cells in the film plane. By contrast, the film thickness is usually limited by practical considerations, such as etching capabilities, film growth technology, etc. Therefore we need to consider what effect the film surfaces have on the previous discussion, where they were ignored. Clearly, near those surfaces the $E$ and $J$ fields will no longer be independent of $x$, and $E_x$ will no longer be uniform. One might guess that, in order to approximate the behavior of an infinitely thick film, the thickness $h$ should be much greater than the heterogeneity size scales. Fortunately, the actual requirement seems to be considerably less stringent. Numerical computations on a square array of perfectly insulating circular cylinders have shown that, although the in-plane resistivity becomes independent of $H$ when $h \rightarrow 0$, already when $h$ is only $1.5$ times greater than the unit-cell size $a$, or $2.5$ times greater than the cylinder diameter $2R$, the magnetoresistance of the film, i.e., the $H$-dependent enhancement of its Ohmic resistivity, has reached at least $2/3$ of the $h=\infty$ value. This finding was corroborated by numerical computations of the detailed current-density distribution in a film made of alternating slabs of two different conducting substances, which showed that $E(r)$ and $J(r)$ deviate from their $h=\infty$ forms only in a very thin boundary layer near the film surfaces. It is also consistent with a simple theoretical argument which shows that the $h=0$ limit is approached only when $Hh$ is much less than the heterogeneity length scales. This makes it reasonable to expect that the $h=\infty$ limit is approached when $Hh$ is much greater than those scales. Thus, a large value of $H = \mu |B|$ makes the requirement on $h$ less stringent.

III. MAGNETORESISTANCE OF SOME SPECIFIC MICROSTRUCTURES WHEN $B$ IS LARGE

In periodic microstructures, the strong-field magnetoresistance exhibits a strong oscillatory dependence on the precise directions of the magnetic field $B$ and of the volume averaged current density $\langle J \rangle$ with respect to the periodic structure. In the past, this was studied mainly using numerical calculations of the bulk effective resistivity tensor $\rho_e$ and of the nonuniform current distribution $J(r)$.

Three dimensional microstructures, as well as two-dimensional or columnar microstructures, were considered in those studies. Using the tools described in Secs. II A and II B, we now proceed to discuss the strong-field magnetoresistance of some columnar microstructures. In particular, because the dual conductivity/
to \( \mathbf{B} \) and \( z \). In that case, the various fields and current densities are uniform in those slabs:

\[
E_{Dz} = \langle E_{Dz} \rangle \Rightarrow J_y = \langle J_y \rangle, \quad (3.1)
\]

\[
E_{Dy} = J_{Dy} = J_z = 0, \quad (3.2)
\]

\[
E_x = \rho_0 H \langle J_y \rangle / p_{host} \Rightarrow J_x = H \left( J_y - \frac{\langle J_y \rangle}{p_{host}} \right) = -H \langle J_y \rangle \frac{1 - p_{host}}{p_{host}}, \quad (3.3)
\]

In a complementary slab, determined as the convex hull or envelope of a row of inclusions along the \( z \) direction, \( E_{Dz} \) and \( J_z \) are nonzero only in the host portions. Their values are determined by the distance \( \zeta(y) \), as explained above.

In the case of a square array of cylindrical inclusions, of radius \( R \) and unit cell size \( a \times a \), inclusion-free slabs as described above are found, for example, when \( \langle \mathbf{J} \rangle || (010) \), if \( 2R < a/\sqrt{1+n^2} \). If, instead, we have a square array of square-cross-section \((b \times b)\) rods, with the same direction of \( \langle \mathbf{J} \rangle \), then the condition for the existence of inclusion-free slabs perpendicular to that direction is \( b < a/(1+n) \). In both cases, \( J_z \) in the host is given by

\[
J_z(y) = \langle J_z \rangle \frac{a \sqrt{1+n^2}}{\zeta(y)}, \quad (3.4)
\]

where, in the case of cylindrical inclusions we have \( y \) is measured from an inclusion center

\[
\zeta(y) = \begin{cases} 
  a - b, & 0 < |y| < b/2, \\
  b, & b/2 < |y| < a/2 
\end{cases}, \quad (3.6)
\]

for \( n = 0 \),

\[
\zeta(y) = \begin{cases} 
  \sqrt{1+n} \left( a - \frac{b}{n} \right), & 0 < |y| < \frac{b(n-1)}{2\sqrt{1+n}}, \\
  \sqrt{1+n} \left( a - \frac{b}{n} + \frac{1}{2} \right) + |y| \frac{1+n^2}{n}, & \frac{b(n-1)}{2\sqrt{1+n}} < |y| < \frac{b(n+1)}{2\sqrt{1+n}}, \quad (3.7)
\end{cases}
\]

\[
\frac{b(n+1)}{2\sqrt{1+n^2}} < |y| < \frac{a}{2\sqrt{1+n^2}}.
\]

In Fig. 2 we show, schematically, the in-plane components of \( \mathbf{E}_D \) and \( \mathbf{J} \) for the case of a square array of cylindrical inclusions when \( \langle \mathbf{J} \rangle || (010) \) and \( \mathbf{B} || (001) \).

The current component \( J_x \) can now be calculated from Eq. (2.8)—it is equal to \( H \) times the spatial fluctuation of \( J_y \). Finally, the current component \( J_z \) can be calculated by first noting that \( \nabla \cdot \mathbf{J} = 0 \). This, along with the fact that \( \mathbf{J} \) is independent of \( x \), leads to

\[
\frac{\partial J_z}{\partial z} = \frac{\partial J_y}{\partial y} \equiv \langle J_y \rangle \frac{a \sqrt{1+n^2}}{\zeta} \frac{d\zeta}{dy}. \quad (3.8)
\]

Integration of this expression in order to get \( J_z \) must proceed carefully, because \( J_z \) has discontinuous jumps at the host-inclusion interface—the detailed calculation is described in the Appendix for the case of a square array of cylinders when \( \langle \mathbf{J} \rangle || (01n) \perp \mathbf{B} \). The result is

\[
J_z(\langle J_y \rangle) = \begin{cases} 
  \frac{a \sqrt{1+n^2}}{(a \sqrt{1+n^2} - 2\sqrt{R^2-y^2})^2} \frac{2y}{\sqrt{R^2-y^2}} \left( z + \frac{a \sqrt{1+n^2}}{2} \right), & 0 < |y| < R, \quad \sqrt{R^2-y^2} < |z| < \frac{a \sqrt{1+n^2}}{2}, \quad z \geq 0, \quad (3.9)
\end{cases}
\]

Note that, in contrast to \( J_x \) and \( J_y \), whose asymptotic forms depend only upon \( y \) in the host, the asymptotic form of \( J_z \) depends on both \( y \) and \( z \). However, since both \( J_y \) and \( J_z \) saturate as \( |H| \to \infty \), the leading term in the total dissipation is due to \( J_x \), which continues to grow indefinitely with increasing \( H \).

From these current distributions we get, using Eq. (2.10), the following results for the asymptotic behavior of the bulk effective in-plane transverse Ohmic resistivity \( \rho_{E}^{(c)} \) (this notation is chosen for consistency with Ref. 1) of a square array of circular cylinders:
\[
\frac{\bar{\rho}_{\perp}^{(e)}}{\rho_0} \cong H^2 \left[ 1 - 2 \sqrt{1 + n^2} \frac{R}{a} - \frac{1}{1 - \pi (R/a)^2} - \frac{\pi}{2} \frac{(1 + n^2)}{\sqrt{a^2 (1 + n^2) - 4R^2}} \arctan \left( \frac{a \sqrt{1 + n^2 - 2R}}{a \sqrt{1 + n^2 - 2R}} \right) \right],
\]

for \(\langle J \rangle \parallel (01n) \perp \mathbf{B}, \quad 2R < a / \sqrt{1 + n^2}, \quad |H| \gg 1,\)

and a square array of square-cross-section rods

\[
\frac{\bar{\rho}_{\perp}^{(e)}}{\rho_0} \cong H^2 \left\{ \begin{aligned}
&\frac{b^3/a^3}{1 - b^2/a^2}, &\text{for } n = 0, \\
&\frac{b(n + 1)b/a - 2n}{a(n - b/a)} - \frac{b^2/a^2}{1 - b^2/a^2} + 2n \ln \frac{n}{n - b/a}, &\text{for } n \geq 1,
\end{aligned} \right.
\]

For \(\langle J \rangle \parallel (01n) \perp \mathbf{B}, \quad b < a / (1 + n), \quad |H| \gg 1.\)

(3.11)

In Fig. 3 we plot the coefficient of the \(H^2\) term versus the inclusion sizes, as given by Eqs. (3.10), (3.11) and by Eqs. (3.13), (3.15) below, for the cases \(n = 0, n = 1\), along with results of numerical computations of \(\bar{\rho}_{\perp}^{(e)}/(\rho_0 H^2)\) for comparison. The agreement is very good except in the case of square-cross-section rods [Fig. 3(b)] when \(n = 1\) and \(b\) approaches \(a\). The reason for the discrepancy is that, while the coefficient of the \(H^2\) term tends to a finite value as \(b \rightarrow a\) [see Eq. (3.15) below], the field-independent contribution to \(\bar{\rho}_{\perp}^{(e)}\) diverges. Thus, in the numerical computation, where the different contributions to \(\bar{\rho}_{\perp}^{(e)}\) are not separated, the field independent term eventually dominates when \(H\) is finite and \(b\) is sufficiently close to \(a\).

**FIG. 3.** (a) Plot (solid lines) of the coefficient of the \(H^2\) term in \(\bar{\rho}_{\perp}^{(e)}/\rho_0\), vs \(R/a\), for a square array (unit-cell size \(a \times a\)) of circular cylinders (radius \(R\)), when \(\langle J \rangle \parallel (010) \perp \mathbf{B}\) [based upon (3.10)] and when \(\langle J \rangle \parallel (011) \perp \mathbf{B}\) [based upon Eq. (3.10) for \(2R < a / \sqrt{2}\), and upon Eq. (3.13) for \(2R > a / \sqrt{2}\)]. Note that the coefficient diverges when \(R/a \rightarrow 0.5\) in both cases. For comparison, we also show (as hexagonal points) results from numerical computations of \(\bar{\rho}_{\perp}^{(e)}/\rho_0\), where we used \(H = 20\), and where we did not try to separate out the \(H^2\) term. (b) Similar plot for a square array of square-cross-section (\(b \times b\)) rods. Note that, in this case, the coefficient of the \(H^2\) term diverges as \(b \rightarrow a\) only in the case when \(\langle J \rangle \parallel (010)\), but not when \(\langle J \rangle \parallel (011)\). The numerical results (square points) appear to diverge even when \(\langle J \rangle \parallel (011)\), because the \(H\)-independent part of \(\bar{\rho}_{\perp}^{(e)}/\rho_0\), which was not separated out, diverges as \(b \rightarrow a\). Insets focus on the behavior at large values of \(R/a\) or \(b/a\), where the contrast between the (010) and (011) results becomes very great. Vertical lines mark the threshold sizes where, for \(\langle J \rangle \parallel (011)\), the inclusion-free slabs disappear.

**FIG. 2.** (a) Schematic pattern of dual currents in transverse transport along the (010) axis of the insulating cylinder array when \(|H| \gg 1\). \(J_{Dz}\) is independent of \(z\), and is uniform in the inclusion-free slabs perpendicular to \(y\) in the host. (b) Distribution of physical current: \(J_z\) is also independent of \(z\) (in the host), and is uniform in the above-mentioned slabs.
If there are no inclusion-free slabs that are perpendicular to \( \mathbf{J} \), then the nonuniformity of \( J_e \) in the host is considerably diminished, resulting in a similarly diminished value for the nonsaturating current component \( J_e \). Consequently, although \( \tilde{\rho}_{\perp}^{(e)} \) still continues to increase as \( H^2 \) for large \( H \), the coefficient of the \( H^2 \) term is much smaller. That is why the angular plot of \( \tilde{\rho}_{\perp}^{(e)} \) versus the direction of \( \mathbf{B} \), which is perpendicular to \( \mathbf{J} \), exhibits a sharp maximum whenever \( \mathbf{B} \) lies in a direction where such slabs exist, unless the slabs are very thin. It should be emphasized that, even along directions of \( \mathbf{B} \) where \( \tilde{\rho}_{\perp}^{(e)} \) has a minimum, it will nevertheless exhibit \( H^2 \) behavior for sufficiently large \( H \). Thus, although numerical computations performed earlier seemed to show that \( \tilde{\rho}_{\perp}^{(e)} \) saturates at those points [see Figs. 12(c), 12(f), and 12(i) of Ref. 1], we now know that the finite values of \( H \) that were used, together with the finite accuracy of the computation, prevented us from resolving the unsaturated \( H^2 \) behavior.  

The criteria for the existence of inclusion-free slabs when \( \mathbf{B}/(0n\hat{1}) \) in a square array of inclusions, namely, the inequalities \( 2R < a/\sqrt{1+n^2} \) (in the case of cylindrical inclusions) and \( b < a(1+n) \) (in the case of square-cross-section, rod-shaped inclusions), can also be obtained by invoking a simple physical picture of the interaction between current distortions produced by neighboring inclusions. In this picture, the region of strong distortion produced by an isolated inclusion has dimensions similar to those of the inclusion in the \( y \) direction, but in the \( z \) direction, along \( \mathbf{B} \), it extends out to a distance that is greater than the inclusion size by a factor \( \mid H \mid \) (see Refs. 14, 15). The interactions between distortions produced by pairs of neighboring inclusions in a periodic array (this is analogous to multiple scattering) are responsible for the maxima and minima in the angular profiles of the components of \( \tilde{\rho}_{\perp}^{(e)} \) versus the direction of \( \mathbf{B} \). But those interactions can only be effective if the elongated region of strong distortion is able to extend from one inclusion to its neighbor along \( \mathbf{B} \) without interference from other inclusions that are closer to it. This geometric shadow picture of the interactions between distortions by different inclusions was first used in Ref. 1 in order to explain the appearance of minima in other components of \( \tilde{\rho}_{\perp}^{(e)} \). Later it was realized that the geometric shadow picture must be supplemented by a more careful consideration of the interaction between two distortion patterns. Approximating that interaction by a simple superposition, it was found that, as in the case of interfering waves, those distortion patterns can either cancel or reinforce each other—see Ref. 13. The present discussion arrived at the same criteria from a different point of view. Instead of concentrating on just a single pair of inclusions, and approximating their interaction by superposing the distortion patterns produced by each one as an isolated inclusion, here we considered the entire periodic array, exploiting the duality transformation and the simplicity of the asymptotic large \( H \) behavior. Our conclusion is that, when either \( R \) or \( b \) decreases through the above mentioned thresholds, associated with \( \mathbf{B}/(0n\hat{1}) \), then the angular plot of \( \tilde{\rho}_{\perp}^{(e)} \) versus the direction of \( \mathbf{B} \) develops a sharp maximum in that direction. Below the threshold, but not above it, the coefficient of the \( H^2 \) term is much larger when \( \mathbf{B}/(0n\hat{1}) \) than when \( \mathbf{B} \) lies along a direction that is even only slightly different, where there are no inclusion-free slabs parallel to \( \mathbf{B} \).

As another example, we consider the case where \( \langle \mathbf{J} \rangle/(011) \) and \( 2R > a/\sqrt{2} \) in a square array of cylindrical inclusions, so that there are no inclusion-free slabs perpendicular to \( \mathbf{J} \). We then get the following results for \( \tilde{\rho}_{\perp}^{(e)} \) (note that, in this case, there is still only one value of \( \tilde{\rho}_{\perp} \) for any \( y \); \( y \) is measured from an inclusion center):

\[
\tilde{\rho}_{\perp}^{(e)}(y) = \begin{cases} 
\frac{a}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \sqrt{y^2 - \frac{a^2}{\sqrt{2}}}, & 0 < y < \frac{a}{\sqrt{2}} - \frac{a}{\sqrt{2}} \times, \\
\frac{a}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \sqrt{y^2 - \frac{a^2}{\sqrt{2}}}, & \frac{a}{\sqrt{2}} - \frac{a}{\sqrt{2}} < y < R, \\
\frac{a}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \sqrt{y^2 - \frac{a^2}{\sqrt{2}}}, & R < y < \frac{a}{\sqrt{2}}. 
\end{cases}
\]

\[
\frac{\tilde{\rho}_{\perp}^{(e)}}{\rho_0} \approx H^2 \left[ \frac{2}{\int_0^1 (1 - \frac{2Ry}{a^2 - x^2})^{1/2} + 2} \int_0^{2R/a - 1/2} \frac{dx}{1 - \frac{2Ry}{a^2 - x^2} - \frac{1}{1 - \pi R^2/a^2}} \right].
\]

The first of these integrals can easily be evaluated explicitly in terms of elementary functions, while the second integral can easily be evaluated numerically. Obviously, the coefficient of the \( H^2 \) term only depends on the ratio \( R/a \). Using both Eqs. (3.10) and (3.13), the values of that coefficient are plotted versus \( R/a \) in Fig. 3(a). Note that, although that coefficient diverges when \( R/a \rightarrow 0.5 \), [because the second integral of Eq. (3.13) diverges in that limit] it has the very small value 0.02395 when \( R/a = 0.4 \). Thus, when \( H = 11 \) the \( H^2 \) term contributes only 2.9 to the value of \( \tilde{\rho}_{\perp}^{(e)}/\rho_0 \). That explains why this term, which leads to nonsaturating behavior at large \( H \), was not resolved in the earlier numerical computations [see Fig. 12(c) of Ref. 1]. Note also that the coefficient of the \( H^2 \) term is a smooth function (i.e., both it and its
first derivative are continuous functions) of $R/a$, even when $2R = a/\sqrt{2}$, at which point the analytic form of that function changes from Eq. (3.10) with $n = 1$ to Eq. (3.13).

In the similar case of a square array of square-cross-section rods, when $(\mathbf{J}) \parallel (011)$ and $2b > a$, again there are no inclusion-free slabs perpendicular to $(\mathbf{J})$. We then get [in this case too, there is only one value of $J_z(y)$ for any $y$, and $y$ is measured from an inclusion center]

$$J_z(y) = \begin{cases} \sqrt{2}(a - b) + 2|y|, & 0 < |y| < \frac{a - b}{\sqrt{2}}, \\ 2\sqrt{2}(a - b), & \frac{a - b}{\sqrt{2}} < |y| < \frac{a}{2\sqrt{2}}, \end{cases}$$

(3.14)

$$\frac{\bar{\rho}_0^{(e)}}{\rho_0} \approx H^2 \left( 2 \ln 2 - \frac{3}{2} + \frac{b/2}{a + b} \right).$$

(3.15)

In Fig. 3(b) we plot the coefficient of the $H^2$ term for this case vs. $b/a$. Again it is a smooth function even at the threshold $2b = a$, where its analytic form changes from (3.11) to (3.15). At this point we would like to note that in this case $\bar{\rho}_0^{(e)}$ is a monotonically increasing function of $b$, whereas for the similar case in a circular-cylinder array $\bar{\rho}_0^{(e)}$ exhibits a maximum near $R/a = 0.32$ and a minimum near $R/a = 0.40$—see Fig. 3(a).

We have calculated the detailed current distribution $\mathbf{J}(y, z)$ for a number of cases where $H$ is large but finite, using numerical methods developed earlier, in order to compare with the asymptotic predictions discussed above. Some of those distributions are shown in Figs. 4 and 5, for $(010) \parallel (\mathbf{J}) \perp \mathbf{B}$ and $(011) \parallel (\mathbf{J}) \perp \mathbf{B}$, respectively. Figure 4(b), which shows $J_x(y, z)$, agrees very well with the asymptotic prediction of Eq. (3.5), using (3.4), while Fig. 4(a), which shows $J_y(y, z)$, agrees very well with Eq. (2.8), and Fig. 4(c), which shows $J_z(y, z)$, agrees very well with Eq. (3.9). The main qualitative features of these agreements are clear even to the naked eye: In the host, $J_x$ is proportional to the spatial fluctuations of $J_z$; in the inclusion-free slabs the value of $J_y$ is equal to the constant $\langle J_x \rangle$, which was taken to be 1; in the other parts of the host $J_y$ depends only on $y$. Note that $J_z$ depends on both $y$ and $z$ in the host, and is of the same order as $J_y$. However, both $J_y$ and $J_z$ make unim-

![FIG. 4. Numerically computed 3D plots (upper) and contour plots (lower) of current density distributions as function of $y$ and $z$, for square array of perfectly insulating circular cylinders when $(\mathbf{J}) \parallel (010) \perp \mathbf{B}$, $H = 20$, and $R/a = 0.4$. (a) $J_x$, (b) $J_y$, (c) $J_z$.](image)

![FIG. 5. Numerically computed 3D plots (upper) and contour plots (lower) of current density distributions as function of $y$ and $z$, for square array of perfectly insulating circular cylinders when $(\mathbf{J}) \parallel (011) \perp \mathbf{B}$ and $H = 20$. (a) $J_y$ for $R/a = 0.4$ (above the threshold), (b) $J_y$ for $R/a = 0.3$ (below the threshold).](image)
portant contributions to the total dissipation, and hence to $\tilde{\rho}_\perp^{(c)}$: At large fields those quantities are governed by $J_x$, which is larger than the spatial fluctuations of $J_y$ by the factor $H = 20$.

In Fig. 5 we show the function $J_x(y, z)$ for two different sizes of the cylindrical inclusions $R/a = 0.4$ [Fig. 5(a)] and $R/a = 0.3$ [Fig. 5(b)], which lie above and below the threshold value $R/a = 1/\sqrt{8} \approx 0.3536$ for that particular direction of $\langle J \rangle / |011\rangle$. Again, the results are in very good agreement with the asymptotic predictions, based on (3.5) for $R/a = 0.3$ and on (3.12) for $R/a = 0.4$. In Fig. 5(b) one again finds the trenches where $J_x = \langle J_y \rangle$ in the inclusion-free slabs. In the host regions outside those slabs in Fig. 5(b), and everywhere in the host in Fig. 5(a), the variation of $J_y$ with $y$ is obviously weak, though $J_y$ is not a constant. Therefore the spatial fluctuations of $J_y$ in the host are much greater when inclusion-free slabs are present, hence $J_x$ is also much greater then, and consequently also the coefficient of the $H^2$ term in $\tilde{\rho}_\perp^{(c)}$. That is why a sharp maximum in the angular plots of $\tilde{\rho}_\perp^{(c)}$ versus the direction of $\mathbf{B}$ for large $H$ always appears in directions where such slabs are found.

In Fig. 6 we show some angular profiles of $\tilde{\rho}_\perp^{(c)}/\rho_0$ versus $\mathbf{B}$, obtained from the numerical computations that we performed, along with some exact asymptotic results, obtained using Eqs. (3.10), (3.11), (3.13), and (3.15). The good agreement between the asymptotic results and the numerical computations increases our confidence in those numerical procedures, which have to be used whenever $H$ is not very large, or when the inclusion shapes are not very simple, or when $\langle J \rangle$ and $\mathbf{B}$ do not lie along a simple, low order lattice axis. Note the appearance of minima when $\mathbf{B} || (01\bar{1})$ in Figs. 6(e) and 6(f), where the inclusion size is above the threshold, and the appearance of sharp maxima in that same direction in Figs. 6(a) and 6(b), where the inclusion size is below the threshold.

**B. Longitudinal resistivity for a collection of parallel, perfectly insulating inclusions**

When $\langle J \rangle || \mathbf{B} \parallel z$, the average dual field $\langle \mathbf{E}_D \rangle || y \perp \mathbf{B}$ points in a direction of low asymptotic dual conductivity in the host. Nevertheless, dual current has to flow in that direction, at the macro scale as well as at the micro scale. However, because the $z$-direction dual conductivity is so high in the host, the local in-plane current flow in the host will choose the $\pm z$ directions whenever possible, or whenever that can make $J_{Dy}$ smaller or more uniform in the host.

Using these considerations to discuss a square array of parallel inclusions (either circular cylinders or square-cross-section rods), we can easily determine the asymptotic dual current distribution, when $\langle J \rangle$ points in any direction from the class $(01n)$ and whenever the inclusions are not too large, so that inclusion-free parallel slabs of host material exist that are perpendicular to $\langle \mathbf{E}_D \rangle$. In such slabs, dual current must flow in the disfavored $y$ direction. In order to minimize the dual dissipation, $J_{Dy}$ will be distributed uniformly in those slabs, therefore $J_{Dy} = \langle J_y \rangle$. By contrast, in the complementary parallel slabs, determined as the convex envelope of a parallel row of inclusions, the in-plane dual current will be confined to those inclusions, where the in-plane dual conductivity is infinite. In those complementary slabs, $E_{Dz}$ vanishes.

When passing through the imaginary boundary between two different slabs, dual current will also flow in the $\pm z$ directions in order to allow $J_{Dz}$ to redistribute itself. It is therefore clear that $E_{Dz}$, and hence $J_z$, will be nonzero only in the inclusion-free slabs, where $J_z = \langle J_z \rangle / \rho_{slab} (\rho_{slab}$ is the volume fraction of those slabs). $E_{Dz}$, and hence $J_z$, will be nonzero only near the boundary between two different slabs, but because $E_{Dz} = \rho_{slab} J_{Dz}/(1 + H^2)$, (note that $E_z = E_{Dz} = 0$, because $\langle J_z \rangle = \langle J_y \rangle = 0$) therefore we expect that $J_z$, and even $J_z = HJ_z$, can be neglected—this is discussed below. Since only $J_z$ is non-negligible, the total dissipation, and hence also the longitudinal bulk effective Ohmic resistivity $\rho^{(c)}_z$, is expected to saturate as $|H| \rightarrow \infty$ at a value given by

$$\frac{\rho^{(c)}_z}{\rho_0} = \frac{\langle W \rangle}{\rho_0 \langle J_y^2 \rangle} = \frac{\langle J_y^2 \rangle}{\rho_{slab}} = \frac{1}{\rho_{slab}}.$$  

In the case of a square array of parallel, cylindrical inclu-

![Graph showing polar plots of $\tilde{\rho}_\perp^{(c)}/\rho_0$ for different inclusions and Hosts](image)
sions of radius $R$ and unit cell size $a \times a$, when $\mathbf{J}$ is directed along the lattice axis $(01n)$ and $2R < a / \sqrt{1+n^2}$, the current density in the inclusion-free slabs is

$$J_z = \frac{\langle J_z \rangle}{1 - 2\sqrt{1+n^2}R/a}.$$  \hspace{1cm} (3.17)

and the saturated asymptotic value of $\rho_1^{(e)}$ is given by

$$\rho_1^{(e)} = \frac{1}{1 - 2\sqrt{1+n^2}R/a}.$$  \hspace{1cm} (3.18)

The current pattern for the case $n = 0$ is shown schematically in Fig. 7. The results for $\rho_1^{(e)}$ compare favorably with numerical computations at large but finite $H$, as can be seen from Fig. 6(c). The minima which appear in the angular dependence of $\rho_1^{(e)}$, when $\mathbf{B}||(01n)$ and $2R < a / \sqrt{1+n^2}$, are due to the fact that, for neighboring directions of $\mathbf{B}$, $\rho_1^{(e)}$ does not saturate as $|H| \to \infty$—see below. Figures 8(a), 8(b), and 8(c) show plots of numerical computations of the three components of $\mathbf{J}(y,z)$ for an insulating cylinders array when $(\mathbf{J})||\mathbf{B}||(010)$ and $H = 20$. The agreement with the asymptotic predictions described above is generally very good. Figure 8(d) shows $J_z(y,z)$ for the much smaller value $H = 1.5$, at which the asymptotic behavior is not expected to be fully developed—compare with Fig. 8(c), which shows $J_z(y,z)$ for $H = 20$. The thin transition layers, which are especially obvious in the plots of $J_x$ and $J_y$ [Figs. 8(a) and 8(b)], are in qualitative agreement with the asymptotic discussion given below. The narrow peaks that appear in all components of $\mathbf{J}$ whenever the inclusion-free slabs touch the cylindrical inclusions indicate a singular behavior associated with the topology of the point contacts. Those peaks are not present in the asymptotic expressions, and they apparently do not contribute to the leading asymptotic behavior of $\rho_1^{(e)}$.

In the case of a square array of parallel rods, with square cross section $b \times b$ and unit cell size $a \times a$, when $(\mathbf{J})||\mathbf{B}||(01n)$ and $b(1+n) < a$, the current density in the inclusion-free slabs is

$$J_z = \frac{\langle J_z \rangle}{1 - (1+n)b/a}, \quad J_x = J_y = 0,$$  \hspace{1cm} (3.19)

and the saturated asymptotic value of $\rho_1^{(e)}$ is given by...
This also compares favorably with numerical computations—see Fig. 6(a). As in the case of the cylinder array, the angular dependence of $\rho_i^{(e)}$ exhibits minima in the (01n) directions because in the neighboring directions $\rho_i^{(e)}$ does not saturate. The local current and field patterns in this case are shown schematically in Fig. 9 for $n = 1$.

We shall see below that when the inclusion size exceeds the above-mentioned thresholds, i.e., when $2R > a/\sqrt{1+n^2}$ for the cylinders or $b > a/(1+n)$ for the square rods, then $\rho_i^{(e)}$ no longer saturates as $|H| \to \infty$, even when $\langle J \rangle \| B \| (01n)$. This is in perfect agreement with a criterion, formulated earlier for the nonappearance of a minimum in that direction in the angular dependence of $\rho_i^{(e)}$, namely [see Eq. (16) of Ref. 1],

$$2R > \frac{a}{\sqrt{1+n^2}}. \quad (3.21)$$

This criterion was originally obtained by considering the simplified geometric shadow picture of the interaction between current distortions produced by neighboring obstacles—see Sec. III A.

Using the same geometric shadow picture for the square rods array, we arrive at the criterion

$$b > \frac{a}{1+n} \quad (3.22)$$

for the nonappearance of a minimum in $\rho_i^{(e)}$ when $B \| (01n)$. Again, this agrees with our present analysis of the magnetotransport behavior.

The situation in the thin transition layer between complementary slabs requires further discussion, which will enable us to estimate its thickness $w$ as well as its contribution to the total dissipation. Due to the need for redistributing $J_{Dz}$ in the transition layer, we can assert that $J_{Dz}$ in that layer must have an order of magnitude given by

$$wJ_{Dz} \sim dJ_{Dy}, \quad (3.23)$$

where $d$ is the distance between neighboring inclusions. Using this estimate, we get

$$E_{Dz} = \rho_0 J_{Dz} \sim \rho_0 J_{Dz} \frac{w}{d}. \quad (3.24)$$

Using the fact that $\langle J_y \rangle = \langle J_z \rangle = 0$, we also get $E_x = E_{Dx} = 0$, and hence [see Eq. (2.16)]

$$J_{Dz} = \frac{1+H^2}{H^2} E_{Dz} \Rightarrow E_{Dz} = \frac{\rho_0 J_{Dz}}{H^2}. \quad (3.25)$$

From $\nabla \times E_D = 0$ we now get, in the transition layer,

$$\frac{\partial E_{Dy}}{\partial z} = \frac{\partial E_{Dz}}{\partial y} \sim \frac{E_{Dz}}{w}. \quad (3.26)$$

This is used to estimate the change in $E_{Dy}, \delta E_{Dy}$, along the transition layer

$$\delta E_{Dy} \approx \frac{d}{wH^2} E_{Dz} \approx \left(\frac{d}{wH}\right)^2 \rho_0 J_{Dz} = \left(\frac{d}{wH}\right)^2 E_{Dy}. \quad (3.27)$$

Clearly, we must have

$$\delta E_{Dy} \approx E_{Dy} \Rightarrow w \approx d/H. \quad (3.28)$$

This estimate for $w$ can now be used to get, in the transition layer,

$$J_y = \frac{E_{Dz}}{\rho_0} J_{Dz} \sim \frac{d}{wH^2} J_{Dy} = \frac{d}{wH^2} \left(\frac{E_{Dy}}{\rho_0}\right)$$

$$= -\frac{d}{wH^2} J_y \ll J_z, \quad \text{for } |H| \gtrsim 1, \quad (3.29)$$

$$J_z = HJ_y \sim \frac{d}{wH} J_z \ll J_z. \quad (3.30)$$

Outside the transition layer, both $J_y$ and $J_z$ are negligible. Using the above estimates, we can calculate the contribution of the transition layer to $\langle J^2_y \rangle$ and $\langle J^2_z \rangle$,

$$\langle J^2_y \rangle \sim \frac{d}{wH^2} \langle J^2_z \rangle \ll \langle J^2_z \rangle, \quad \text{for } |H| \gtrsim 1, \quad (3.31)$$

$$\langle J^2_z \rangle \sim \frac{d}{wH^2} \langle J^2_z \rangle \ll \langle J^2_z \rangle, \quad \text{for } |H| \gtrsim 1. \quad (3.32)$$
We conclude that the leading contribution to the total dissipation, and hence to \( \rho_{\parallel}^{(e)} \), is entirely due to \( J_z \), as assumed above, justifying the general asymptotic result (3.16) whenever \( p_{\text{slab}} > 0 \). Moreover, Eq. (3.28) suggests that the width \( w \) of the transition layer is in fact of order \( 1/|H| \).

When the inclusions are large enough so that there are no inclusion-free slabs perpendicular to \( (E_D) \) (or parallel to \( (J) \)), the situation is drastically different. Dual current can now avoid having to flow through the host in the disfavored \( y \) direction. It will flow in that direction only inside the perfectly conducting dual inclusions. Between neighboring inclusions the dual current will flow mostly in the \( \pm z \) directions, with the possible exception of some thin transition layers, such as we already encountered above.

Perhaps the simplest case where such behavior appears is in a square array of parallel, square-cross-section rods, when \( (J)/(011) \), and when the rod size \( b \times b \) and the unit cell size \( a \times a \) satisfy \( 2b > a \). In that case, the microscopic configuration of dual currents is as shown in Fig. 10(a). The dual current in the host flows mostly in the \( \pm z \) directions, and is uniform throughout each of the parallelogram-shaped regions outlined in the gaps between neighboring rods. Outside those regions \( J_D \approx 0 \). The in-plane components of \( E_D \) have similarly uniform values in those regions—see Fig. 10(b),

\[
E_{Dz} = \pm \frac{\langle E_{Dy} \rangle}{2(1-b/a)}, \quad E_{Dy} = \frac{\langle E_{Dy} \rangle}{2(1-b/a)}.
\]

This means that the planar component of \( E_D \) inside the parallelogram-shaped regions is always perpendicular to the straddling inclusion surfaces, and its magnitude there is \( \langle E_{Dy} \rangle/\sqrt{2(1-b/a)} \). Note that, when \( H = 0 \) and if the gap width between neighboring inclusions \( a-b \) is much smaller than \( a \), then we would expect these same values of \( E_{Dz} \) and \( E_{Dy} \) to be valid over the entire volume of that gap. In contrast, when \( |H| \gg 1 \), these values hold only in the smaller parallelogram-shaped regions within that gap, but their validity is not restricted to small values of \( a-b \); they are valid whenever \( 2b > a \). Outside the parallelograms \( E_{Dz} = 0 \), because \( J_{Dz} = 0 \) there. However, \( E_{Dy} \) does not vanish there—see Figs. 10(a) and 10(b).

The components of \( J \) in those parallelograms are easily obtained from the above results,

\[
J_z = \frac{\langle J_z \rangle}{2(1-b/a)}, \quad J_y = \pm \frac{\langle J_z \rangle}{2(1-b/a)}, \quad J_x = \pm \frac{H\langle J_z \rangle}{2(1-b/a)}.
\]

Outside the parallelograms, \( J_y = J_z = 0 \), but \( J_z \) is nonzero and saturated. Note that, since \( J_y = 0 \) in the host, \( J_z \) is nonzero and nonsaturating. Thus the longitudinal resistivity is also nonsaturating. Its asymptotic behavior is given by

\[
\rho_{\parallel}^{(e)} = \frac{\langle W \rangle}{\rho_0 \langle J_z \rangle^2} \approx H^2 \left( \frac{\langle J_z \rangle}{\langle J_z \rangle^2} \right)^2 = H^2 \frac{a-1}{b} \frac{1}{b} \frac{1}{a}.
\]

This is in good agreement with numerical computations of \( \rho_{\parallel}^{(e)} \)—see Fig. 6(b).

In the case of a square array of cylindrical inclusions, \( |B(01n)| \) and \( 2R > a\sqrt{1+n^2} \), the asymptotic current distribution is more complicated, though it can still be found by invoking the principles enunciated above. Again, dual current in the host flows mostly in the \( \pm z \) directions, but its magnitude is continuously nonsaturating. Nevertheless, because \( J_{Dz} = 0 \), the magnitude of \( J_{D} \) is constant along every current flow line between neighboring inclusions. Thus, the field component \( E_{Dz} = \rho_0 J_{Dz} / (1+H^2) \) is also nonsaturating, but is also constant along such a flow line, and its saturated value is proportional to the distance \( z \), and \( J_{Dz} \) covers in getting from one inclusion to its nearest neighbor along a particular flow line. The dual current configurations for the cases \( n = 1 \) and \( n = 2 \) are shown qualitatively in Fig. 11. Because \( E_{Dz} \) is nonsaturating and saturated in the host, \( J_z = E_{Dz} / \rho_0 \) also has those properties, and consequently \( J_x = HJ_z / \rho_0 \) is nonzero and does not saturate. Thus \( \rho_{\parallel}^{(e)} / \rho_0 \approx \langle J_z \rangle^2 / \langle J_z \rangle^2 \times H^2 \), i.e., \( \rho_{\parallel}^{(e)} \) does not saturate.
Singling out the case \( n = 1 \) for a more detailed discussion, we recall that, in this case, \( \ell_z(y) \) is given by Eq. (3.12). However, in the present case only the middle line of that equation is important, since the first and third lines correspond to distances between perfectly conducting dual inclusions that are at the same potential. In the range of \( y \) corresponding to the middle line of Eq. (3.12) we find

\[
E_{Dz}(y) = \pm \langle E_{Dy} \rangle \frac{a}{\sqrt{2} \ell_z(y)},
\]

while for the other values of \( y \) we get \( J_{Dz} = E_{Dz} = J_{y} = J_{x} = 0 \). From these expressions, it is easy to obtain, after some algebra, the following asymptotic expression for the longitudinal resistivity (recall that \( \langle J_{z} \rangle = 0 \)).

Note that this is the same expression as the middle term of Eq. (3.13). This integral was evaluated numerically, and the results compare favorably with numerical computations of \( \rho_{0}^{(e)} \)—see Fig. 6(g).

Note that, in spite of the fact that \( J_{z} \) and \( J_{y} \) have similar magnitudes, we did not need to know the asymptotic distribution of \( J_{z} \) or \( E_{Dy} \), which is rather complicated, in order to get the asymptotic result for \( \rho_{0}^{(e)} \) in Eqs. (3.35) and (3.39).

We would like to emphasize the fact that the system behavior is qualitatively different when there exist inclusion-free slabs parallel to \( \mathbf{J} \) and when such slabs are absent. Thus, when \( \mathbf{J} \) and \( \mathbf{B} \) are both directed along the \((01n)\) axis of a square array of perfectly insulating inclusions, the local current pattern changes drastically when the inclusion sizes cross the threshold values, given by \( 2R = a/\sqrt{1+n^2} \) for cylindrical inclusions and by \( b = a/(1+n) \) for square-cross-section rod-shaped inclusions (compare Figs. 9 and 10). As a consequence of this, the asymptotic behavior of \( \rho_{0}^{(e)} \) also changes drastically, from saturated to unsaturated \( H^2 \) behavior, when the inclusion sizes increase through these thresholds. This is also shown by the asymptotic curves of Fig. 12(d). Nevertheless, we note that the coefficient of the \( H^2 \) term is continuous at the threshold—it increases continuously, starting at 0, above the threshold. We would also like to note that indications of such drastic changes in asymptotic behavior with increasing size of the inclusions were already present in numerical results published earlier—see Figs. 12(b), 12(e), and 12(h) of Ref. 1.

In the case of a disordered collection of insulating inclusions, we cannot get any closed form expressions. However, it is clear that the dual current in the host will flow mostly along the \( \pm z \) directions, and its magnitude will be comparable to \( \langle E_{Dy} \rangle / \rho_{0} \). The same kind of reasoning that we used to discuss periodic arrays of large inclusions then leads to the conclusion that \( \rho_{0}^{(e)} \) never saturates, but continues to grow as \( H^2 \) for large \( H \). Of course, if the microstructure is isotropic in the \( y,z \) plane, then the angular profile of \( \rho_{0}^{(e)} \) will also be isotropic.

C. Longitudinal resistivity for a collection of parallel, perfectly conducting inclusions

Because the inclusions are perfect conductors, we again use \( \rho_{0} \) and \( H \) to denote the appropriate finite parameters of the host, which is a normal, free-electron-like conductor. In the case under discussion, we have \( \langle \mathbf{J} \rangle \parallel \mathbf{B} \parallel \langle \mathbf{E}_{D} \rangle \parallel y \), and the dual inclusions have zero conductivities in the \( y,z \) plane [see Eq. (2.19)]. Dual current will have to flow locally in the \( y \) direction. However, because the dual conductivity of the host in the \( z \) direction is very high when \( |H| \gg 1 \), there will be local flows of \( J_{Dz} \) that will redistribute the current in such a way that, for every value of \( y \), both \( J_{Dy} \) and \( E_{Dy} \) in the host are independent of \( z \) in between any pair of adjacent inclusions. It follows that

\[
J_{z} = \frac{E_{z}}{\rho_{0}} = -J_{Dy} \propto \frac{1}{\ell_{z}(y)}.
\]

where \( \ell_{z}(y) \) is the distance along \( z \) between neighboring inclusions at a given value of \( y \). The proportionality coefficient in this relation depends on the potential difference between the two perfectly conducting inclusions.

In order to estimate the other components of \( \mathbf{J} \), we first note that the redistribution role of \( J_{Dz} \) means that, wherever \( \ell_{z}(y) \) is a continuous function, \( J_{Dy} \) is of the same order of magnitude as \( J_{Dz} \). As before, we shall denote such a relationship symbolically by \( J_{Dz} \approx J_{Dy} \). The components of \( \mathbf{E}_{D} \) in the host satisfy

\[
E_{Dx} = E_{x} = 0,
\]

\[
E_{Dy} = \rho_{0} J_{z} = \rho_{0} J_{Dy},
\]

\[
E_{Dz} = \rho_{0} J_{Dz}/(1 + H^2).
\]

Therefore
\[ J_y = \frac{1}{H^2} E_{Dz} \left( \zeta \right) - J_z \left( H \right), \quad \text{for } |H| > 1, \quad (3.44) \]

\[ J_y = H J_z = \frac{J_z}{H}, \quad \text{for } |H| > 1, \quad (3.45) \]

and the leading contribution to the dissipation \( W \) comes from \( J_z \).

When \( \gamma_y(y) \) is discontinuous, the redistribution of \( J_{Dy} \) requires some very large values of \( J_{Dz} \) in a thin transition layer, whose thickness \( w \) must satisfy

\[ w J_{Dz} \sim d J_{Dy}, \quad (3.46) \]

where \( d \) is the distance between neighboring inclusions. The entire chain of reasoning that lead from Eq. (3.23) to Eqs. (3.29) and (3.30) can now be repeated, leading to the same conclusion, namely, that even the large values of \( J_y \) and \( J_z \) in the transition layers make only a negligible contribution to the total dissipation at large \( H \).

Because the only important current component at large \( H \) is \( J_z \), which is saturated, the average dissipation, and consequently also \( \rho_0^{(c)} \), are saturated. This is true for all microstructures, periodic and disordered alike, and for all directions of \( B \). In the case of a simple periodic array, such as the square arrays of cylinders or rods discussed earlier, and \( B \) along a lattice axis of sufficiently low order, simple closed form asymptotic expressions can be written for \( \gamma_y(y) \) and \( J_z \), and consequently also for \( \rho_0^{(c)}/\rho_0 \).

For example, in a square array of cylindrical inclusions, when \( \langle J \rangle \| B \|_{(01)} \| z \) and \( 2 R < a/\sqrt{1 + n^2} \), so that there exist inclusion-free slabs along \( z \), we can use the expressions for \( \gamma_z(y) \) of Eq. (3.5) to get (the constant \( J_z \) determined by \( \langle J_z \rangle \); \( y \) is measured from an inclusion center; note that \( \langle J_z \rangle \) denotes a volume average over the entire volume, but with \( J_z \) replaced by 0 inside the perfectly conducting inclusions)

\[ J_z(y) = J_z(0) \left\{ \begin{array}{ll}
\frac{a \sqrt{1 + n^2}}{a \sqrt{1 + n^2} - 2 \sqrt{R^2 - y^2}} & 0 < |y| < R, \\
1 & R < |y| < \frac{a}{2 \sqrt{1 + n^2}}.
\end{array} \right. \quad (3.47) \]

\[ \frac{\rho_1^{(c)}}{\rho_0} = \frac{\langle J_z \rangle_{\text{host}}}{\langle J_z \rangle^2} = \left[ 1 - 2 \sqrt{1 + n^2} \frac{R}{a} \frac{\pi}{a} \left( 1 + n^2 \right) \right. \right.
\[ + \frac{2(a + n^2)}{\sqrt{a^4(1 + n^2) - 4R^2}} \arctan \left( \frac{a \sqrt{1 + n^2} + 2R}{a \sqrt{1 + n^2} - 2R} \right) \right]^{-1}. \quad (3.48) \]

In a square array of square-cross-section rods, when \( \langle J \rangle \| B \|_{(01)} \| z \) and \( b < a/(1 + n) \), so that there exist inclusion-free slabs along \( z \), we can similarly use the expressions for \( \gamma_z(y) \) of Eqs. (3.6), (3.7) to get

\[ \left\{ \begin{array}{ll}
1 - \frac{b}{a}, & n = 0, \\
\frac{b}{1 - \frac{b}{a} - \frac{b}{a}}, & n = 1,
\end{array} \right. (3.49) \]

which appears whatever the direction of \( B \), will also prevail for a disordered array of parallel, perfectly conducting inclusions.

D. Transverse resistivity for a collection of parallel, perfectly conducting inclusions

In this case we have \( \langle J \rangle \| y \perp B \| \langle E_D \rangle \| z \). If there exist inclusion-free slabs that are parallel to \( z \), through which dual current can flow from end to end of the system along the \( z \) direction, then \( J_{Dz} \) will be nonzero and uniform only in those slabs (because fanning out into the host regions in the complementary slabs would require nonvanishing values of \( J_{Dz} \), which would exact an unacceptable price in total dual dissipation when \( |H| \to \infty \), where \( E_{Dz} = \langle E_{Dz} \rangle \). The components of \( J_p \) are given, in those slabs, by (recall that \( E_{Dx} = E_z = 0 \))

It is interesting to note that the discussion in this subsection, and also some of the results, are similar to those of Sec. III A, where we considered the transverse resistivity of a collection of perfectly insulating columnar inclusions. This is a reflection of the duality symmetry that exists in these systems. Nevertheless, the bulk effective resistivity has a drastically different dependence on \( H \) in the two cases.

In Fig. 13 we show the current and field patterns for a square array of rods, when \( \langle J \rangle \| y \perp B \| \langle E_D \rangle \| z \), so that there exist inclusion-free slabs along \( \langle J \rangle \). Note the similarity of Figs. 2 and 13, if we transpose the original pattern and the dual pattern. In Figs. 14(b) and 14(f) we show angular plots of \( \rho_1^{(c)} \) versus the direction of \( B \), obtained using numerical calculations, along with some special points calculated using the asymptotic expressions (3.48) and (3.49). Evidently the agreement is excellent. The saturating behavior of \( \rho_1^{(c)} \),

\[ \rho_1^{(c)} = \left\{ \begin{array}{ll}
1 - \frac{b}{a}, & n = 0, \\
\frac{b}{1 - \frac{b}{a} - \frac{b}{a}}, & n = 1,
\end{array} \right. (3.49) \]
while the components of \( \mathbf{J} \) are given by

\[
J_z = -\frac{1}{\rho_0} E_{Dy}, \quad J_y = 0, \quad J_x = \frac{1}{\rho_0} E_{Dz},
\]

(3.50)

(3.51)

(3.52)

(3.53)

FIG. 12. Plots (lines) of \( \tilde{\rho}_{i}^{(c)}/\rho_0 \) [(a) and (c)] and \( \rho_{0,0}^{(c)}/\rho_0 \) [(b) and (d)] from exact asymptotic expressions for square arrays of inclusions, vs the inclusion size \( \xi \), which represents \( R/a \) in the case of circular cylinders, \( b/(2a) \) in the case of square rods, using \( H = 20 \). The points show results of numerical calculations using the same parameters. Results are shown for perfectly insulating inclusions (empty points) as well as for perfectly conducting inclusions (filled points), where “perfect conductivity” in fact means \( \rho_{0,0,0} / \rho_0 \) inclusion \( = 10^3 \) or \( 10^5 \). (a) and (b) show results for \( \langle \mathbf{J} \rangle \) along the principal axis (010), with the full lines representing cylinder arrays and the dashed lines representing rod arrays (note that sometimes the different lines lie on top of each other), while (c) and (d) show results for \( \langle \mathbf{J} \rangle \) along the 45° lattice axis (011). Hexagon-shaped points and asterisks correspond to cylinders, square-shaped points and crosses correspond to rods. Note that (c) and (d) are plotted using semilogarithmic scales (natural logarithms, and not base-10 logarithms). Note also that, in contrast to the asymptotic results, the numerical results contain all the contributions, and not just the leading power of \( H \) term—which is the cause for the discrepancies in (c) when \( b \) approaches 0 or \( a \), and which is also the cause for the discrepancies in both (c) and (d) near the threshold \( b = a/2 \). The discontinuities in asymptotic behavior appear at that threshold because the coefficient of the \( H^2 \) term goes to zero there: in (c) as \( b \) increases, and in (d) as \( b \) decreases.

In a complementary slab, determined as the convex envelope of a row of inclusions along \( z \), the dual field component \( E_{Dz} \), and the current components \( J_z, J_x, J_y \), are nonzero only inside the perfectly conducting inclusions. In a thin transition layer between the two types of slabs, there is a large current component \( J_z \), which serves to redistribute \( J_x \) and \( J_y \). In order to estimate the importance of \( J_z \) in this layer, we note that \( w \) is of the order of the transition layer thickness, \( d \) is

FIG. 13. Local current and field pattern (schematic) for perfectly conducting rod-shaped inclusions when (010)|\langle \mathbf{J} \rangle|\mathbf{B} : (a) \( J_D \), (b) \( \mathbf{J} \).

FIG. 14. Lines show polar plots of \( \tilde{\rho}_{i}^{(c)}/\rho_0 \) and \( \rho_{0,0}^{(c)}/\rho_0 \) vs the direction of \( \mathbf{B} \) for square arrays of perfectly insulating and perfectly conducting inclusions, of cylindrical shape as well as of square-rod shape, obtained from numerical calculations using \( H = 20 \), \( R/a = 0.3 \), and \( b/a = 0.3 \). Large points show asymptotic values calculated from the appropriate expressions, using the same parameters. The large discrepancy in (d) when \( \mathbf{B} |(02\bar{1}) \) is due to the fact that \( b/a = 0.3 \) is very close to the threshold value for that direction, namely, \( b/a = 1/3 \).
of the order of the distance between inclusions)
\[
\frac{\partial E_{Dz}}{\partial z} = \frac{\partial E_{Dz}}{\partial y} \sim \frac{E_{Dz}}{w} \Rightarrow E_{Dy} \sim \frac{d}{w} E_{Dz} \gg E_{Dz} \tag{3.54}
\]
there. Therefore the average dual dissipation satisfies
\[
\rho_0 \langle W_D \rangle = (\langle E^2_{Dy, host} \rangle + (1 + H^2) \langle E^2_{Dz, host} \rangle) \sim \frac{w}{d} \left( \frac{d}{w} \langle E_{Dz} \rangle \right)^2 + p_{slab} (1 + H^2) \langle E_{Dz} \rangle^2.
\tag{3.55}
\]
Because the second term is independent of \(w\), we can obviously minimize \(\langle W_D \rangle\) by having
\[
\frac{d}{w} \ll H^2. \tag{3.56}
\]
In the original problem we have, in the transition layer,
\[
J_z = - \frac{1}{\rho_0} E_{Dy} \sim \frac{d}{w} \frac{E_{Dz}}{\rho_0} = \frac{d}{w} J_y \gg J_y.
\tag{3.57}
\]
Therefore
\[
\langle J_z^2 \rangle \sim \frac{w}{d} \left( \frac{d}{w} \langle J_y \rangle \right)^2 \ll H^2 \langle J_y \rangle^2 \sim \langle J_y^2 \rangle,
\tag{3.58}
\]
and finally
\[
\frac{\rho_{slab}^{(e)}}{\rho_0} = \frac{\langle W \rangle}{\rho_0 \langle J_z^2 \rangle} \equiv \frac{\langle J_z^2 \rangle}{\langle J_y^2 \rangle} \equiv H^2 p_{slab}.
\tag{3.59}
\]
In the case of a square array of cylindrical inclusions, when \(\langle J \rangle || (01n) \perp B\) and \(2R < a/\sqrt{1 + n^2}\). \(R\) is the cylinder radius, \(a \times a\) is the unit-cell size) we get
\[
\frac{\rho_{slab}^{(e)}}{\rho_0} \equiv H^2 [1 - 2 \sqrt{1 + n^2} R/a], \quad \text{for} \quad 2R < a/\sqrt{1 + n^2}.
\tag{3.60}
\]
In the case of a square array of square-cross-section rods, when \(\langle J \rangle || (01n) \perp B\) and \(b < a/\langle 1 + n \rangle\), \((b \times b\) is the rod cross section) we get
\[
\frac{\rho_{slab}^{(e)}}{\rho_0} \equiv H^2 [1 - (1 + n) b/a], \quad \text{for} \quad b < a/\langle 1 + n \rangle.
\tag{3.61}
\]
The field and current patterns for the latter case, when \(n = 0\), are shown schematically in Fig. 15. Again, note the similarity between this and Fig. 7, if we transpose the original pattern and the dual pattern. Angular plots of \(\rho_{slab}^{(e)}/\rho_0\) versus the direction of \(B\) are shown in Figs. 14(a) and 14(c)—the lines show results of numerical calculations while the points show asymptotic predictions of Eqs. (3.60) and (3.61).

If there are no inclusion-free slabs perpendicular to \(\langle J \rangle\), then it is not necessary to have any local nonzero values of \(E_{Dz}\) in the host—the entire dual potential drop across the system will be achieved with the help of nonzero values of \(E_{Dz}\) in the inclusions, where the dual in-plane conductivity vanishes. The system always selects this kind of field configutation because it minimizes the dissipation in the dual host, where the conductivity along the \(z\) direction is very high. In this case, the local in-plane dual field is a zigzagging field in the \(y\) direction, such that \(E_{Dy} \sim \langle E_{Dy} \rangle\) but \(\langle E_{Dy} \rangle = 0—\text{see, e.g.,} \quad \text{Fig. 16.} \quad \text{The local dual currents along} \quad y \quad \text{and} \quad z \quad \text{are comparable} \quad J_{Di} \sim J_{Dy}, \quad \text{hence the local values of} \quad E_{Dz} \quad \text{in the host satisfy}
\[
E_{Dz} \sim \frac{\rho_0}{1 + H^2} J_{Di} - \frac{\rho_0}{H^2} J_{Dy} = \frac{E_{Dy}}{H^2} \ll E_{Dy}.
\tag{3.62}
\]
This translates into the following results for the components of \(J\) in the host:
\[
J_z = \frac{E_{Dz}}{\rho_0} = - J_{Dy}, \tag{3.63}
\]
\[
J_y = \frac{E_{Dz}}{\rho_0} \frac{E_{Dy}}{H^2} = - \frac{J_z}{H} \ll J_z, \tag{3.64}
\]
\[
J_z = H J_y + \frac{J_z}{H} \ll J_z. \tag{3.65}
\]
Thus, the average dissipation \(\langle W \rangle\) and the in-plane transverse resistivity \(\rho_{slab}^{(e)}\) depend only on \(J_z\), to leading order in \(H\). The distribution of \(J_z\) can be found from \(E_{Dy}\) or \(J_{Dy}\), and they are all saturated.

This is the usual state of affairs when the columnar microstructure is disordered. It is also the usual state of affairs even when the microstructure is periodic but \(\langle J \rangle\) points in an
FIG. 16. Local current and field pattern (schematic) for perfectly conducting rod-shaped inclusions when \((011)\|\langle J\rangle\): (a) \(J_O\), (b) \(E_O\), (c) \(E\), (d) \(J\).

arbitrary direction. Thus, the usual behavior of \(\tilde{\rho}_{\perp}^{(e)}\) is saturation at large \(H\). The existence of inclusion-free slabs perpendicular to \(\langle J\rangle\), which leads to \(\tilde{\rho}_{\perp}^{(e)} \approx H^2\) at large \(H\), only occurs if \(\langle J\rangle\) lies along a low-order lattice axis and if the inclusions are small enough. E.g., in the case of a square array of cylinders or rods, such slabs exist when \(\langle J\rangle\|\langle 01n\rangle\) if the inclusion sizes satisfy the inequalities of Eqs. (3.60) or (3.61), respectively.

When those inequalities are not satisfied, it is possible in these cases to calculate closed form expressions for the asymptotic saturated value of \(\tilde{\rho}_{\perp}^{(e)}\). Since \(J_z\) is the only non-zero component of \(J\) in the host, its value must be constant along any flow line between two inclusions. The constant value of \(E_z = \rho_0 J_z\) along that line will be inversely proportional to the distance \(\sqrt{z}\) along it between the two inclusions. In the case of a square array of square-shaped rods, when \(\langle J\rangle\|\langle 011\rangle\) and \(b > a/2\), the various field and current patterns are shown, qualitatively, in Fig. 16. We find that \(J_z\) is non-zero in the host only in the parallelogram-shaped regions shown in that figure, where it has the constant value

\[
J_z = \pm \frac{(J_y)}{2b/a - 1}, \tag{3.66}
\]

and therefore

\[
\tilde{\rho}_{\perp}^{(e)} = \left\langle \frac{J_z^2}{\langle J_y \rangle^2} \right\rangle \approx \left\langle \frac{1 - b}{a} \right\rangle \approx \left\langle \frac{b}{2} \right\rangle. \tag{3.67}
\]

Again, note the similarity between Figs. 10 and 16, when we transpose the original pattern and the dual pattern.

The asymptotic results for \(\tilde{\rho}_{\perp}^{(e)}\) versus \(b/(2a)\) are compared with numerical calculation results in Figs. 12(a) and 12(c). Although the overall trends are in agreement, there are quantitative discrepancies: The numerical results remain significantly below the asymptotic predictions even when \(H\) is made quite large, especially when \(\langle J\rangle\|\langle 011\rangle\). At present we do not understand this. Perhaps it is a sign that, in contrast with some of our previous estimates, the neglect of the thin transition layers is not entirely justified in this case. Clearly, this discrepancy needs to be studied further.

We would again like to emphasize that both the local current pattern and \(\tilde{\rho}_{\perp}^{(e)}\) undergo a drastic change when the inclusion-free slabs are eliminated by making the inclusions larger when \(\langle J\rangle\|\langle 01n\rangle\), for \(n \geq 1\). This can be seen by comparing the current patterns of Figs. 15 and 16, and by considering the asymptotic plot of \(\tilde{\rho}_{\perp}^{(e)}\) versus the inclusion sizes shown in Fig. 12(c). In this case, the unsaturated \(\sim H^2\) asymptotic behavior appears when the inclusion size is below the threshold, and it switches over to saturated asymptotic behavior above that threshold. As we saw in Sec. III B, here too the coefficient of the \(H^2\) term is continuous at the threshold: It tends to 0 as the threshold is approached from below. The drastic change in behavior of \(\tilde{\rho}_{\perp}^{(e)}\) for columnar arrays of perfectly conducting inclusions has recently been found also in numerical simulations of an appropriate discrete network model.  

The fact that only in special directions, and only when the inclusions are not too large, will \(\tilde{\rho}_{\perp}^{(e)}\) exhibit unsaturated behavior, \(i.e., \tilde{\rho}_{\perp}^{(e)} \sim H^2\) for large \(H\) means that the angular dependence of \(\tilde{\rho}_{\perp}^{(e)}\) on the direction of \(\langle J\rangle\) or \(B\) will have very sharp maxima in some of those directions, when the inclusion sizes are small enough. These peaks will usually be narrower and more pronounced than in the case of inclusions that are perfectly insulating, because in the latter case the usual situation is that \(\tilde{\rho}_{\perp}^{(e)} \sim H^2\) for any direction of \(\langle J\rangle\). This expectation is in agreement with our numerical computations—compare Figs. 14(a) and 14(e) with Figs. 14(c) and 14(g).

IV. SUMMARY AND DISCUSSION

The magnetotransport properties of composite conductors with a periodic columnar microstructure were found to exhibit a number of surprising features, including lack of saturation and very strong anisotropy at large \(B\). Those features are predicted to appear in a regime where the local response is entirely describable in terms of classical linear transport theory. This means that one might expect to observe such behavior in appropriately fabricated materials, without having to exert the extremely low temperatures and cleanliness with respect to impurities and defects that are usually needed in order to observe macroscopic or mesoscopic quantum effects. More experimental work is needed to test our predictions, especially for the case where the inclusions are perfect conductors. Such work might also lead to ideas for practical exploitation of the magnetotransport properties of thin conducting or semiconducting films with a periodic array of columnar inclusions. Of special significance in this regard are the drastic changes in local current patterns and asymptotic behavior of the resistivities that occur when the inclusion-free slabs that are parallel to \(B\) shrink to zero with increasing size of the inclusions. Those drastic changes occur in \(\rho_\parallel\) for the case of inclusions that are perfectly insulating, and in \(\rho_\perp^{(e)}\) for the case of inclusions that are perfectly conducting, and were described in Secs. III B and III D, respectively. They are also emphasized in Figs. 12(c) and 12(d), where they
appear as infinite jumps in \( \rho^{(c)}_1 \) and \( \rho^{(c)}_0 \) at the threshold \( b = a/2 \). (The infinite jumps are the technical manifestation of the fact that the leading asymptotic behavior switches from \( H^2 \) to \( H^0 \), or vice versa, at the threshold). Figure 14 summarizes some of the main properties of the angular dependence of \( \bar{\rho}^{(c)}_1 \) and \( \bar{\rho}^{(c)}_0 \) on the direction of \( B \) for square arrays of parallel inclusions that are either perfect insulators or perfect conductors, where the inclusion shapes are either circular cylinders or square rods. The lines in that figure were produced by “brute force” numerical computations, while the points represent asymptotic results, obtained from the theory developed in this article. Thus, Fig. 14 also summarizes the main features of the agreement between the asymptotic theory and those computations. The agreement is usually very good. When a large discrepancy exists, as in the case of the \( B(001) \) points in Fig. 14(d), this is usually well understood. In this case it is due to the fact that the inclusion size is very close to the threshold for that direction, namely, \( a/3 \), at which point \( \rho^{(c)}_1 \) for an array of insulating inclusions undergoes a drastic change from saturating to nonsaturating dependence on \( H \). The numerical results shown in Figs. 14(a) and 14(e), along with the discussion at the end of Sec. III D, lead to the conclusion that the strongest high-field anisotropy appears when one considers the \( \bar{\rho}^{(c)}_1 \) component of the resistivity tensor for an array of perfectly conducting inclusions, and that the circular-cylinder inclusions are better in that respect than the square-rod inclusions.

While this article was being written, we learned of the recent work by Fisher and Stroud,\(^4 \) in which they have also observed a drastic change in the behavior of a discrete network model for a square array of perfectly conducting cylindrical inclusions when \( \langle J \rangle || (011) \perp B \) and \( 2R = a/\sqrt{2} \). By using the duality transformation, we were able to go beyond that discussion and provide a quantitative description of this change, and also to identify and discuss similar drastic changes in \( \rho^{(c)}_1 \) for a periodic array of perfectly insulating inclusions.

As we already indicated at some points in this article, the duality transformation has useful implications also for columnar composites with a disordered microstructure. We will explore some of those implications in a forthcoming article.

Even from a cursory perusal of the asymptotic expressions obtained in this work, it is clear that some simple relations exist between asymptotic macroscopic magnetotransport properties of arrays of perfectly insulating inclusions and macroscopic magnetotransport properties of arrays of perfectly conducting inclusions, when the microstructures are the same. E.g., from Eqs. (3.16) and (3.59), and also from Eqs. (3.35) and (3.67), we immediately get that the product of \( \rho^{(c)}_1 / \rho_0 \) for an array of perfectly insulating inclusions and \( \bar{\rho}^{(c)}_1 / \rho_0 \) for an array of perfectly conducting inclusions is simply \( H^2 \).

\[
\frac{\rho^{(c)}_1}{\rho_0} \left( \frac{\bar{\rho}^{(c)}_1}{\rho_0} \right) \approx H^2.
\]

This relation has to do with the fact that the perfectly insulating and perfectly conducting inclusions are somewhat related to each other by duality. Those relations will be explored in greater detail elsewhere.

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**APPENDIX**

In order to calculate \( J_y(y,z) \) for the case of a square array of perfectly insulating cylinders, when \( \langle J \rangle || (01n) \perp B \), we first write the form of \( J_y(y) \) over an entire unit cell, using Eqs. (3.4) and (3.5):

\[
\frac{J_y}{\langle J \rangle} \approx \begin{cases} 
\frac{a \sqrt{1+n^2}}{a \sqrt{1+n^2} - 2 \sqrt{R^2-y^2}} & 0 < |y| < R, \quad \sqrt{R^2-y^2} < |z| < \frac{a \sqrt{1+n^2}}{2}, \\
1 & R < |y| < a, \quad 0 < |z| < \frac{a \sqrt{1+n^2}}{2}, \\
0 & \text{elsewhere}; \quad 0 < |y| < R, \quad 0 < |z| < \sqrt{R^2-y^2}.
\end{cases}
\]

We then take the derivative of this function to get

\[
-\frac{\partial}{\partial y} \left( \frac{J_y}{\langle J \rangle} \right) \approx -\frac{a \sqrt{1+n^2}}{a \sqrt{1+n^2} - 2 \sqrt{R^2-y^2}} \left[ \delta(y - \sqrt{R^2-y^2}) + \delta(y + \sqrt{R^2-y^2}) \right] \begin{cases} 
\frac{a \sqrt{1+n^2}}{a \sqrt{1+n^2} - 2 \sqrt{R^2-y^2}} & 0 < |y| < R, \quad \sqrt{R^2-y^2} < |z| < \frac{a \sqrt{1+n^2}}{2}, \\
0 & R < |y| < a, \quad 0 < |z| < \frac{a \sqrt{1+n^2}}{2}, \\
0 & \text{elsewhere}; \quad 0 < |y| < R, \quad 0 < |z| < \sqrt{R^2-y^2}.
\end{cases}
\]

We now use
in order to evaluate \( J_z(y, z) \) by integration of (A2) over \( z \). For the middle regime in Eq. (A2), namely, \(|y| > R\), we thus get that \( J_z \) is independent of \( z \). Consequently, since \( E_z = \rho_d J_z \) and the integral \( \int dz \, E_z \) must vanish, it follows that \( J_z = 0 \) when \(|y| > R\). For \(|y| < R\), \( J_z \) obviously vanishes when also \(|z| < \sqrt{R^2 - y^2}\), which is inside the perfectly insulating inclusion. Outside the inclusion, where \(|z| > \sqrt{R^2 - y^2}\) [the first regime in Eq. (A2)], integration over \( z \) leads to Eq. (3.9).

\[
\frac{\partial J_z}{\partial z} = -\frac{\partial J_y}{\partial y}
\]  

(A3)

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